ON CHROMATIC NUMBERS OF SPACE-TIMES: OPEN PROBLEMS

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Chromatic number: reminder. A *chromatic number* $\chi(S)$ of a metric space S is defined as the smallest number of colors in a coloring of S in which no two points of the same color are unit distance apart; see, e.g., (Soifer 2009) and references therein.

Space-times: reminder. What will happen if we define a similar quantity for *space-times*, e.g., for pseudo-Euclidean (Minkowski) spaces $\mathrm{R}^{1,n}$ in which the distance $d(A, B)$ between the two points $A = (a_0, a_1, \ldots, a_n)$ and $B = (b_0, b_1, \ldots, b_n)$ is defined as

$$d(A, B) = \sqrt{(a_0 - b_0)^2 - (a_1 - b_1)^2 - \ldots - (a_n - b_n)^2}$$

(see, e.g., (Naber 2003)). In such spaces, we have two different notions of a distance: the *temporal* distance

$$\tau(A, B) = \sqrt{(a_0 - b_0)^2 - (a_1 - b_1)^2 - \ldots - (a_n - b_n)^2}$$

which is defined when $(a_0 - b_0)^2 - (a_1 - b_1)^2 - \ldots - (a_n - b_n)^2 \geq 0$, and the *spatial* distance

$$\rho(A, B) = \sqrt{(a_1 - b_1)^2 + \ldots + (a_n - b_n)^2 - (a_0 - b_0)^2}$$

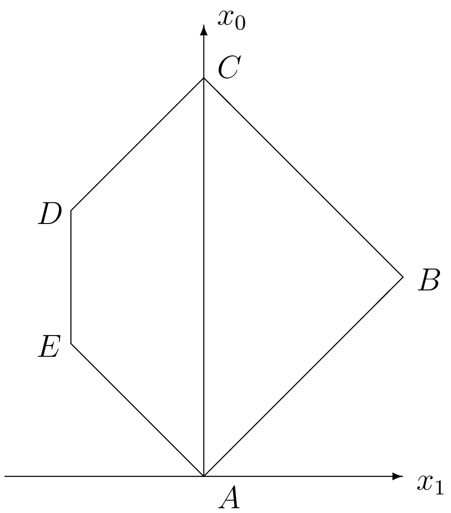
which is defined when $(a_0 - b_0)^2 - (a_1 - b_1)^2 - \ldots - (a_n - b_n)^2 \leq 0$.

Chromatic number of a space-time: definitions. In line with (Koshelev 1998), we can define three different versions of the chromatic number for a space-time $S = \mathrm{R}^{1,n}$:

- $\chi_\tau(\mathrm{R}^{1,n})$, the smallest number of colors in a coloring of $\mathrm{R}^{1,n}$ in which no two points A and B with $\tau(A, B) = 1$ have the same color;

- $\chi_\rho(\mathrm{R}^{1,n})$, the smallest number of colors in a coloring of $\mathrm{R}^{1,n}$ in which no two points A and B with $\rho(A,B) = 1$ have the same color;
- $\chi_{\tau\rho}(\mathrm{R}^{1,n})$, the smallest number of colors in a coloring of $\mathrm{R}^{1,n}$ in which no two points A and B with $\tau(A,B) = 1$ or $\rho(A,B) = 1$ have the same color.

Chromatic number of a space-time: a simple bound. What can we same about these values? It is easy to see that for the Minkowski plane, $\chi_\tau(\mathrm{R}^{1,1}) \geq 3$. Indeed, for every $h > 1$, it is not possible to color the following pentagonal configuration in 2 colors:



with $A = (0,0)$, $B = (0.5+h, \sqrt{(0.5+h)^2 - 1})$, $C = (1+2h, 0)$, $D = (h+1, \sqrt{h^2-1})$, and $E = (h, \sqrt{h^2-1})$.

First open problem. What is the value of $\chi_\tau(\mathrm{R}^{1,1})$? Is it finite?

Chromatic number of a space-time: possible meaning. The temporal distance τ measures the proper time between the two events.

A possible meaning of $\chi_\tau(\mathrm{R}^{1,n})$ is, e.g., that if we assume that we measure time in number of generations, then the events A and B corresponding to two immediately following generations should always be of different color. Informally, in the Newtonian space-time, even though musical (and other) tastes change from generation to generation, it is still possible that there are pendulum-type oscillations between two tastes, with grandchildren going back to what parents rebeled against.

In the relativistic case, as we have seen, two oscillating tastes are not enough, so how many do we need?

An alternative, more physical meaning: let us assume that some system-changing process takes a proper time τ_0 – hence, in the appropriate measuring units, the proper time 1. What is the smallest number of states for which the two objects at the same point in space-time are always in the same state?

What about ρ? Case of Minkowski plane. In the Minkowski plane, the transformation $x_0 \leftrightarrow x_1$ swaps τ and ρ, so $\chi_\rho(\mathrm{R}^{1,1}) = \chi_\tau(\mathrm{R}^{1,1})$ (and hence, $\chi_\rho(\mathrm{R}^{1,1}) \geq 3$).

What about ρ? General case. In general, by mapping each point $(a_1, \ldots, a_n)$ into $(0, a_1, \ldots, a_n)$, we can embed R^n into $\mathrm{R}^{1,n}$ so that the standard Euclidean distance turns into the spatial distance ρ.

Thus, every c-colors coloring of $\mathrm{R}^{1,n}$ in which no two points of the same color have $\rho(A, B) = 1$ leads to a coloring of R^n in $\leq c$ colors – hence $\chi(\mathrm{R}^n) \leq \chi_\rho(\mathrm{R}^{1,n})$.

Case of Busemann space-time metrics. What if instead of the Minkowski definitions of $\tau(A, B)$ and $\rho(A, B)$, we take a real number $\alpha \geq 1$ and consider more general *Busemann metrics* (Busemann 1967, Kreinovich 1998)

$$\tau_\alpha(A, B) = ((a_0 - b_0)^\alpha - (a_1 - b_1)^\alpha - \ldots - (a_n - b_n)^\alpha)^{1/\alpha}$$

(defined when $(a_0 - b_0)^\alpha - (a_1 - b_1)^\alpha - \ldots - (a_n - b_n)^\alpha \geq 0$) and

$$\rho_\alpha(A, B) = ((a_1 - b_1)^\alpha + \ldots + (a_n - b_n)^\alpha - (a_0 - b_0)^\alpha)^{1/\alpha}$$

(defined when $(a_0 - b_0)^\alpha - (a_1 - b_1)^\alpha - \ldots - (a_n - b_n)^\alpha \leq 0$)?

Case of general pseudo-Euclidean spaces. What if we consider a general pseudo-Euclidean space $\mathrm{R}^{m,n}$ (not necessarily related to space-time) in which the distance $d(A, B)$ between the two points $A = (a_1, a_2, \ldots, a_{m+n})$ and $B = (b_1, b_2, \ldots, b_{m+n})$ is defined as

$$d(A, B) = \sqrt{D(A, B)},$$

where

$$D(A, B) \stackrel{\text{def}}{=} (a_1 - b_1)^2 + \ldots + (a_m - b_m)^2 -$$
$$(a_{m+1} - b_{m+1})^2 - \ldots - (a_{m+n} - b_{m+n})^2.$$

In such spaces, we can also consider:

- the *temporal* distance $\tau(A, B) = \sqrt{D(A, B)}$ which is defined when $D(A.B) \geq 0$, and
- the *spatial* distance $\rho(A, B) = \sqrt{-D(A, B)}$, which is defined when $D(A, B) \leq 0$.

Here also natural embeddings lead to $\chi(\mathrm{R}^m) \leq \chi_\tau(\mathrm{R}^{m,n})$ and $\chi(\mathrm{R}^n) \leq \chi_\rho(\mathrm{R}^{m,n})$.

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