

Towards a Natural Proof of Metrization Theorem for Space-Times

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Abstract—In the early 1920s, Pavel Urysohn proved his famous lemma (sometimes referred to as “first non-trivial result of point set topology”). Among other applications, this lemma was instrumental in proving that under reasonable conditions, every topological space can be metrized.

A few years before that, in 1919, a complex mathematical theory was experimentally proven to be extremely useful in the description of real world phenomena: namely, during a solar eclipse, General Relativity theory – that uses pseudo-Riemann spaces to describe space-time – has been (spectacularly) experimentally confirmed. Motivated by this success, Urysohn started working on an extension of his lemma and of the metrization theorem to (causality)-ordered topological spaces and corresponding pseudo-metrics. After Urysohn’s early death in 1924, this activity was continued in Russia by his student Vadim Efremovich, Efremovich’s student Revolt Pimenov, and by Pimenov’s students (and also by H. Busemann in the US and by E. Kronheimer and R. Penrose in the UK). By the 1970s, reasonably general space-time versions of Uryson’s lemma and metrization theorem have been proven.

However, the proofs of these 1970s results are not natural. Since one of the main objectives of this activity is to come up with useful applications to physics, we definitely desire natural versions of these proofs. In this paper, we show that fuzzy logic leads to such natural proofs.

I. INTRODUCTION

Urysohn’s lemma. In the early 1920s, a mathematician Pavel Urysohn proved his famous lemma (sometimes referred to as “first non-trivial result of point set topology”). This lemma deals with *normal* topological spaces, i.e., spaces in which every two disjoint closed sets have disjoint open neighborhoods; see, e.g., [2]. As the very term “normal” indicates, most usual topological space are normal, including the n -dimensional Euclidean space.

Urysohn’s lemma states that if X is a normal topological space, and A and B are disjoint closed sets in X , then there exists a continuous function $f : X \rightarrow [0, 1]$ for which $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

Resulting metrization theorem. Urysohn’s lemma has many interesting applications. Among other applications, this lemma was instrumental in proving that under reasonable conditions, every topological space can be metrized.

Specifically, from this lemma, we can easily conclude that every normal space X with countable base is *metrizable*, i.e., there exist a *metric* – a function $\rho : X \times X \rightarrow R_0^+$ for which the following three conditions are satisfied

$$\rho(a, b) = 0 \Leftrightarrow a = b;$$

$$\rho(a, b) = \rho(b, a);$$

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$$\rho(a, c) \leq \rho(a, b) + \rho(b, c);$$

and for which the original topology on X coincides with the topology generated by the open balls

$$B_r(x) = \{y : \rho(x, y) < r\}.$$

Comment. It is worth mentioning that the normality condition is too strong for metrizability: actually, it is sufficient to require that the space is:

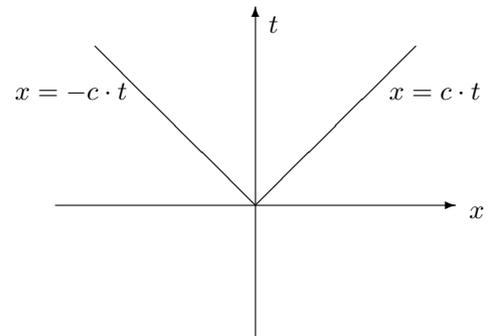
- *regular*, i.e., for every closed set A and every point $b \notin A$ can be separated by disjoint open neighborhoods, and
- *Hausdorff*, i.e., that every two different points have disjoint open neighborhoods.

Space-time geometry and how it inspired Urysohn. A few years before Urysohn’s lemma, in 1919, a complex mathematical theory was experimentally proven to be extremely useful in the description of real world phenomena. Specifically, during a solar eclipse, General Relativity theory – that uses pseudo-Riemann spaces to describe space-time – has been (spectacularly) experimentally confirmed; see, e.g., [9].

From the mathematical viewpoint, the basic structure behind space-time geometry is not simply a topological space, but a topological space with an *order* $a \preceq b$ whose physical meaning is that the event a can causally influence the event b .

For example, in the simplest case of the Special Relativity theory, the event $a = (a_0, a_1, a_2, a_3)$ can influence the event $b = (b_0, b_1, b_2, b_3)$ if we can get from the spatial point (a_1, a_2, a_3) at the moment a_0 to the point (b_1, b_2, b_3) at the moment $b_0 > a_0$ which traveling with a speed which is smaller than or equal to the speed of light c :

$$\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2} \leq c \cdot (b_0 - a_0).$$



Motivated by this practical usefulness of ordered topological spaces, Urysohn started working on an extension of his lemma and of the metrization theorem to (causality-)ordered topological spaces and corresponding pseudo-metrics.

Space-time metrization after Urysohn. P. S. Urysohn did not have time to work on the space-time extension of his results, since he died in 1924 at an early age of 26.

After Urysohn's early death, this activity was continued in Russia by his student Vadim Efremovich, by Efremovich's student Revolt Pimenov, and by Pimenov's students – and also by H. Busemann in the US and by E. Kronheimer and R. Penrose in the UK [1], [4], [12] (see also [7]).

This research actively used the general theory of ordered topological spaces; see, e.g., [10].

By the 1970s, reasonably general space-time versions of Uryson's lemma and metrization theorem have been proven; see, e.g., [5], [6].

Space-time metrization results: a challenge. One of the main objectives of the space-time metrization activity is to come up with useful applications to physics.

From this viewpoint, we definitely need naturally provable versions of these theorems. Alas, the original 1970s proofs of space-time metrization results are not natural. It is therefore necessary to make them natural.

What we do in this paper. In this paper, we follow the ideas of L. A. Zadeh on applying fuzzy to causality (see, e.g., [13]), and show that fuzzy logic indeed leads to such natural proofs.

II. KNOWN SPACE-TIME METRIZATION RESULTS: REMINDER

Causality relation: the original description. The current formalization of space-time geometry start with a transitive relation $a \preceq b$ on a topological space X .

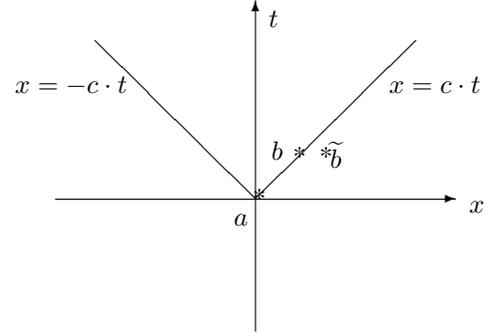
The physical meaning of this relation is causality – that an event a can influence the event b . This meaning explains transitivity requirement: if a can influence b and b can influence c , this means that a can therefore (indirectly) influence the event c .

Need for a more practice-oriented definition. On the theoretical level, the causality relation \preceq is all we need to know about the geometry of space-time.

However, from the practical viewpoint, we face an additional problem – that measurements are never 100% accurate and therefore, we cannot locate events exactly. When we are trying to locate an event a in space and time, then, due to measurement uncertainty, the resulting location \tilde{a} is only approximately equal to the actual one: $\tilde{a} \approx a$.

From this viewpoint, when we observe that an event a influences the event b , we record it as a relation between the corresponding approximations – i.e., we conclude that $\tilde{a} \preceq \tilde{b}$. However, this may be a wrong conclusion: for example, if an event b is at the border of the future cone $F_a \stackrel{\text{def}}{=} \{b : a \preceq b\}$ of the event a , then

- we have $a \preceq b$, but
- the approximate location \tilde{b} may be outside the cone, so the conclusion $a \preceq \tilde{b}$ is wrong.



Kinematic causality: a practice-oriented causality relation. To take into account measurement uncertainty, researchers use a different causality relation $a \prec b$, meaning that every event in some small neighborhood of b causally follows a , i.e., that b belongs to the interior $\text{Int}(F_a)$ of the future cone F_a .

In the simplest space-time of special relativity, this means that we are excluding the border of the future cones (that corresponds to influencing by photons and other particles traveling at a speed of light c) and only allow causality by particle whose speed is smaller than c . The motion of such particles is known as *kinematics*, hence this new practice-oriented causality relation is called *kinematic causality*.

These definition implies, e.g., that the kinematic causality relation is transitive, as well as several other reasonable properties. These properties lead to the following formal definition of the kinematic causality relation.

Resulting definition of kinematic causality. A relation \prec is called a *kinematic causality* if it is transitive and satisfies the following properties:

$$\begin{aligned} a \not\prec a; \quad \forall a \exists \underline{a}, \bar{a} (\underline{a} \prec a \prec \bar{a}); \quad a \prec b \Rightarrow \exists c (a \prec c \prec b); \\ a \prec b, c \Rightarrow \exists d (a \prec d \prec b, c); \\ b, c \prec a \Rightarrow \exists d (b, c \prec d \prec a). \end{aligned}$$

Topology and causality can be defined in terms of kinematic causality relation. We started our description with a pre-ordered topological space X with a causality relation \preceq . Based on topology and on the causality relation, we defined the kinematic causality \prec .

It turns out that in many reasonable cases, it is sufficient to know the kinematic causality relation \prec . Based on this relation, we can uniquely reconstruct both the topology and the original causality relation \preceq .

Indeed, as a topology, we can take a so-called *Alexandrov topology* in which intervals

$$(a, b) \stackrel{\text{def}}{=} \{c : a \prec c \prec b\}$$

form the base.

Once the topology is defined, we can now describe causality as

$$a \preceq b \stackrel{\text{def}}{=} b \in \overline{a^+},$$

where $a^+ \stackrel{\text{def}}{=} \{b : a \prec b\}$ and \overline{S} denotes the closure of the set S .

Comment. In principle, we can use a dual definition $a \preceq b \stackrel{\text{def}}{=} a \in \overline{b^-}$, where $b^- \stackrel{\text{def}}{=} \{c : c \prec b\}$. To make sure that these two definitions lead to the same result, the following additional property is usually required:

$$b \in \overline{a^+} \Leftrightarrow a \in \overline{b^-}.$$

Towards a space-time analog of a metric. Traditional metric is defined as a function $\rho : X \times X \rightarrow R_0^+$ for which the following properties are satisfied:

$$\rho(a, b) = 0 \Leftrightarrow a = b;$$

$$\rho(a, b) = \rho(b, a);$$

$$\rho(a, c) \leq \rho(a, b) + \rho(b, c).$$

The usual physical meaning of this definition is that $\rho(a, b)$ is the length of the shortest path between a and b . This meaning leads to a natural explanation for the triangle inequality $\rho(a, c) \leq \rho(a, b) + \rho(b, c)$. Indeed, the shortest path from a to b (of length $\rho(a, b)$) can be combined with the shortest path from b to c (of length $\rho(b, c)$) into a single combined path from a to c of length $\rho(a, b) + \rho(b, c)$. Thus, the length $\rho(a, c)$ of the shortest possible path between a and c must be smaller than or equal to this combined length: $\rho(a, c) \leq \rho(a, b) + \rho(b, c)$.

In space-time, we do not directly measure distances and lengths. The only thing we directly measure is (proper) time along a path. So, in space-time geometry, we talk about times and not lengths.

It is well known that if we travel with a speed close to the speed of light, then the proper travel time (i.e., the time measured by a clock that travels with us) goes to 0. Thus, in space-time, the smallest time does not make sense: it is always 0. What makes sense is the *largest* time. In view of this, we can define a “kinematic metric” $\tau(a, b)$ as the longest (= proper) time along the path from event a to event b .

Of course, such a path is only possible if a kinematically precedes b , i.e., if $a \prec b$.

If $a \prec b$ and $b \prec c$, then the longest path from a to b (of length $\tau(a, b)$) can be combined with the longest path from b to c (of length $\tau(b, c)$) into a single combined path from a to c of length $\tau(a, b) + \tau(b, c)$. Thus, the length $\tau(a, c)$ of the longest possible path between a and c must be larger than or equal to this combined length: $\tau(a, c) \geq \tau(a, b) + \tau(b, c)$. This inequality is sometimes called the *anti-triangle inequality*.

These two properties constitute a formal definition of a kinematic metric.

Definition of a kinematic metric. By a *kinematic metric* on a X with a kinematic causality relation \prec , we mean a function $\tau : X \times X \rightarrow R_0^+$ that satisfies the following two properties:

$$\tau(a, b) > 0 \Leftrightarrow a \prec b;$$

$$a \prec b \prec c \Rightarrow \tau(a, c) \geq \tau(a, b) + \tau(b, c).$$

Space-time analogue of Urysohn’s lemma. The main condition under which the space-time analog of Urysohn’s lemma is proven is that the space X is *separable*, i.e., there exists a countable dense set $\{x_1, x_2, \dots, x_n, \dots\}$.

The lemma states that if \prec is a kinematic causality relation on a separable space X , and $a \prec b$, then there exists a continuous \preceq -monotonic function

$$f_{(a,b)} : X \rightarrow [0, 1]$$

for which:

- $f_{[a,b]}(x) = 0$ for all x for which $a \not\prec x$, and
- $f_{[a,b]}(x) = 1$ for all x for which $b \preceq x$.

This lemma is *similar* to the original Urysohn’s lemma, because it proves the existence of a function $f_{(a,b)}$ that separates two disjoint closed sets:

- the complement $-a^+$ to the set and
- the set b^+ .

The new statement is *different* from the original Urysohn’s lemma, because:

- first, it only considers special closed sets, and
- second, in contrast to the original Urysohn’s lemma, the new lemma also requires that the separating function f be monotonic.

Space-time analogue of the metrization theorem. If X is a separable topological space with a kinematic causality relation \prec , then there exists a continuous kinematic metric τ which generates the corresponding kinematic causality relation \prec — in the sense that $a \prec b \Leftrightarrow \tau(a, b) > 0$.

Comment. Since, as we have mentioned, the kinematic causality relation \prec also generates the topology, we can conclude that the kinematic metric τ also generates the corresponding topology.

How the (non-constructive) space-time metrization theorem is proved: two more lemmas. In the existing proofs, we first prove the following two lemmas:

- that for every x , there exists a \preceq -monotonic function

$$f_x : X \rightarrow [0, 1]$$

for which $f_x(b) > 0 \Leftrightarrow x \prec b$; and

- that for every x , there exists a \preceq -decreasing function

$$g_x : X \rightarrow [0, 1]$$

for which $g_x(a) > 0 \Leftrightarrow a \prec x$.

III. TOWARDS A NATURAL PROOF FOR THE SPACE-TIME METRIZATION THEOREM

Why fuzzy logic. We start with a crisp causality relation $a \prec b$ describing whether an event a can influence an event b .

We would like to transform it into the numerical “degree of causality” $\tau(a, b)$. This is exactly one of the tasks which which fuzzy logic started: how to transform and commonsense expert statements (difficult to process by a computer) into numerical degrees – degrees that computers can easily process. It is therefore reasonable to apply fuzzy logic methodology to come up with an appropriate function $\tau(a, b)$.

Towards a fuzzy interpretation of space-time ideas. Fuzzy logic starts when instead of knowing the exact value of a quantity x , we only have expert estimates that describes its approximate value.

The simplest case of such an expert estimate of a quantity is when we only have a lower bound \underline{x} and an upper bound \bar{x} for this quantity. In this case, the only information that we have about the quantity x is that $\underline{x} \leq x \leq \bar{x}$, i.e., in mathematical terms, that the value x belongs to the *interval*

$$[\underline{x}, \bar{x}] \stackrel{\text{def}}{=} \{x : \underline{x} \leq x \leq \bar{x}\}.$$

Similarly, in space-time, we may not know the exact spatial and temporal coordinates of a given event x . Instead, we may know an event \underline{x} that influenced x (i.e., that causally precedes x), and an event \bar{x} that was influenced by x (i.e., that causally follows from x).

This situation is typical, e.g., in estimating the date x of an ancient manuscript:

- if the manuscript cites some previous book with a known date \underline{x} , we can conclude that $\underline{x} \leq x$;
- on the other hand, if another manuscript with a known date \bar{x} cites this one, we can conclude that $x \leq \bar{x}$.

In general, once we know two events \underline{x} and \bar{x} , we can therefore conclude that $\underline{x} \leq x \leq \bar{x}$, i.e., that the event x belongs to the *interval*

$$[\underline{x}, \bar{x}] \stackrel{\text{def}}{=} \{x : \underline{x} \leq x \leq \bar{x}\}.$$

Towards a natural fuzzy interpretation of the space-time analogue of Urysohn’s lemma. When we have exactly known events e and x , then we have two options:

- either e can influence x , i.e., e casually precedes x ,
- or e cannot influence x , i.e., e does not casually precedes x .

However, as we have mentioned, in practice, we rarely know the exact event e ; instead, we know the interval $[a, b]$ that contains this event e . Once we know this interval $[a, b]$, then, for a given event x , what can we conclude about the possibility of e causally influencing x ?

- If $b \leq x$, then from $e \leq b$ we can conclude (since causality is transitive) that $e \leq x$.

- If $a \not\leq x$, then we can conclude that $e \not\leq x$; indeed, if we had $e \leq x$, then from $a \leq e$ and transitivity, we would be able to conclude that $a \leq x$, while we know that $a \not\leq x$.

In all other cases, we are not sure whether $e \leq x$ or $e \not\leq x$, it depends on where exactly the event e is in the given interval $[a, b]$. From the viewpoint of the traditional (crisp) mathematics, all we can conclude in these cases is that both options $e \leq x$ and $e \not\leq x$ are possible.

However, from the physical viewpoint, for different events x , we have different degrees of possibility that $e \leq x$:

- If an event x is such that only a small portion of events from $[a, b]$ precede x , then it is not very probable that e precedes x , so our degree of possibility that e precedes x is small.
- On the other hand, if an event x is such that only a small portion of events from $[a, b]$ do not precede x , then it is very probable that the actual (unknown) e precedes x , so our degree of possibility that e precedes x is close to absolutely certainty (i.e., close to 1).

This degree of possibility $f_{[a,b]}(x)$ that $e \leq x$ has the properties that $f_{[a,b]}(x) = 1$ for all x for which $b \leq x$ and $f_{[a,b]}(x) = 0$ for all x for which $a \not\leq x$.

If $x \leq x'$ and $e \leq x$, then, of course, $e \leq x'$. Thus, our degree of possibility that $e \leq x'$ is larger (or equal) than the degree that $e \leq x$. In mathematical terms, this means that the function $f_{[a,b]}(x)$ is \leq -monotonic.

When we change x slightly, our degree $f_{[a,b]}(x)$ should also change only slightly. In mathematical terms, this means that the function $f_{[a,b]}$ should be *continuous*.

Thus, the desired degree $f_{[a,b]}(x)$ is a continuous \leq -monotonic function $f_{[a,b]} : X \rightarrow [0, 1]$ for which:

- $f_{[a,b]}(x) = 0$ for all x for which $a \not\leq x$, and
- $f_{[a,b]}(x) = 1$ for all x for which $b \leq x$.

This is exactly the function whose existence is proven in the space-time analogue of Urysohn’s lemma. Thus, it is natural to interpret the function $f_{[a,b]}(x)$ from this lemma as a degree with which the (exactly known) x causally follows from an event e about which we only know that e belongs to the interval $[a, b]$.

The original proof of the space-time metrization theorem uses the space-time analogue of the Urysohn’s lemma. Let us therefore start our analysis by providing a fuzzy interpretation for this lemma.

This lemma proves the existence of a function $f_{[a,b]}$ from the space-time X into the interval $[0, 1]$. Formally, this means that this function $f_{[a,b]}$ is thus a membership function, i.e., a fuzzy set. However, to provide a natural way of dealing with this lemma, we need to provide a commonsense interpretation, i.e., a commonsense property corresponding to this fuzzy set.

By the formulation of the Urysohn’s lemma, we have two events a and b such that a causally precedes b . In

For the resulting function, we have $f_{[a,b]}(x) = 0$ for all x for which $a \not\leq x$ and $f_{[a,b]}(x) = 1$ for all x for which $b \leq x$. Thus, the corresponding property:

- is absolutely satisfied (with degree of satisfaction 1) for all the events that follow b , and
- is absolutely not satisfied (with degree of satisfaction 0) for all the events that do not causally follow from a .

IV. TOWARDS A NATURAL PROOF OF THE LEMMAS

Without losing generality, let us concentrate on the first lemma. The lemmas are similar, so let us concentrate on the first one.

Formulation of the first lemma: reminder. For every x , there exists a \preceq -monotonic function

$$f_x : X \rightarrow [0, 1]$$

for which $f_x(b) > 0 \Leftrightarrow x \prec b$.

Main idea. Suppose that we are analyzing the consequences of an event x . How can we check whether an event b causally follows from x ? A natural way to do that is if b is influenced by some signal emitted at the moment x .

Ideally, we should consider signals emitted exactly at x . In practice, of course, there is always a delay between the decision to emit the signal and the actual emission. We do not know the exact time of the emission event; however, we know the upper bound y_1 ($x \prec y_1$). With this signal, the influence is confirmed if b follows some (unknown) event from the interval $[x, y_1]$.

By using more and more accurate technologies, we can make this delay smaller and smaller. At first, we get an emission event with a lower bound $y_2 \succ y_1$ that is closer to x than y_1 : $x \prec y_2 \prec y_1$, then we get an event with an even closer upper bound event y_3 , etc. Thus, we get a decreasing sequence $x \prec \dots \prec y_3 \prec y_2 \prec y_1$ that converges to x .

An event x precedes b if there exists an emitting event i whose signal is detected at b , i.e., for which $y \prec b$ for some $y \in [x, y_i]$.

Let us now use fuzzy logic techniques to transform assign, to each event b , a degree to which this statement is satisfied.

Additional idea. From the purely mathematical viewpoint, the above statement is perfectly correct. However, let us recall that the index i describes the efficiency of the corresponding emitters: the larger i , the more accurate (and thus, the more sophisticated) these emitters must be.

From the practical viewpoint, if the corresponding i is too large, this means that we need too sophisticated a technology – a technology that may be not be available for the next hundreds of years – to actually detect that $x \prec b$. Thus, a more practical formulation of the above statement is as follows: $a \prec x$ if and only if

$$\exists i ((i \text{ is not too large}) \& (y \prec b \text{ for some } y \in [x, y_i])).$$

Let us use the simplest fuzzy translations. To transform the above formula into numerical degrees, we need to describe:

- the degree of belief that i is not too large,

- the degree of belief that $y \prec b$ for some $y \in [x, y_i]$, and
- t-norm (“and”) and t-conorm (“or”) operations that will enable us to form the degree of a composite statement $A \& B$ or $A \vee B$ from the degrees of the corresponding statements A and B .

So far, we know the degree of belief that $y \prec b$ for some $y \in [x, y_i]$: it is the function $f_{[x, y_i]}(b)$.

Which t-norms and t-conorms shall we select? One can see, from the following proof, that all selections will work. So, to make our proof as simple as possible, let us select t-norm and t-conorm which are as simple to analyze as possible. Specifically, we will use the algebraic product $a \& b = a \cdot b$ and the sum $a \vee b = a + b$ (to be more precise, we should use $\min(a + b, 1)$, but if we restrict ourselves to sufficiently small values, we can ignore the 1 part).

The quantifier $\exists i A_i$ is nothing else but an infinite “or”-statement: $A_1 \vee A_2 \vee \dots$, so we can use the t-conorm $+$ to estimate its truth value.

The only thing that remains is to describe the degree of belief $N(i)$ that i is not too large.

How to describe the degree of belief that i is not too large.

To describe this degree of belief, let us use the reasonable property that if i is not large and j is not too large, then their sum $i + j$ should also be not too large. By equating the degrees of belief $N(i) \& N(j) = N(i) \cdot N(j)$ and $N(i + j)$ of the statements

$$(i \text{ is not too large}) \text{ and } (j \text{ is not too large})$$

and

$$i + j \text{ is not too large,}$$

we conclude that $N(i + j) = N(i) \cdot N(j)$. Thus, we get

$$N(2) = N(1) \cdot N(1) = N(1)^2,$$

$$N(3) = N(2) \cdot N(1) = N(1)^3$$

and, in general, $N(i) = c^i$ for some constant $c = N(1) < 1$.

One can see that the proof works for all possible values $c < 1$, so let us choose the simplest possible value. In the computer, all the numbers are represented in binary, so the easier value to use is $c = 1/2$. For this value, we get $N(i) = (1/2)^i = 2^{-i}$.

Resulting formula. The degree that i is not too large is equal to 2^{-i} , the degree that $y \prec b$ for some $y \in [x, y_i]$ is equal to $f_{[x, y_i]}(b)$, so the degree with which

$$(i \text{ is not too large}) \& (y \prec b \text{ for some } y \in [x, y_i]))$$

is equal to the product of these two values, i.e., to

$$2^{-i} \cdot f_{[x, y_i]}(b).$$

Thus, our degree of belief in an infinite or-statement

$$\exists i ((i \text{ is not too large}) \& (y \prec b \text{ for some } y \in [x, y_i]))$$

is equal to

$$f_x(b) = \sum_{i=1}^{\infty} 2^{-i} \cdot f_{[x, y_i]}(b).$$

Resulting proof. Once we have this expression, it is now relatively easy to prove that this expression is indeed \preceq -monotonic and that for this expression, $f_x(b) > 0$ if and only if $x \prec b$.

Indeed, monotonicity follows from the monotonicity of all the terms $f_{[x,y_i]}(b)$.

Similarly, if $f_x(b) > 0$, this means that $f_{[x,y_i]}(b) > 0$ for some i . By the properties of the function $f_{[x,y_i]}$, this means that $x \prec b$ – because $f_{[x,y_i]}(b) = 0$ for all b for which $x \not\prec b$.

Vice versa, let us assume that $x \prec b$. This means that x belongs to a set $b^- = \{y : y \prec b\}$ (“past cone of b ”), a set which is open in the Alexandrov topology. In other words, this set b^- is an open neighborhood of x . Since $y_i \rightarrow x$, we conclude that starting from i , all the points i will also belong to this same neighborhood, i.e., we will have $y_i \prec b$. By the properties of the function $f_{[x,y_i]}$, this means that for this i , we have $f_{[x,y_i]}(b) = 1$ and thus, $\sum_{i=1}^{\infty} 2^{-i} \cdot f_{[x,y_i]}(b) > 0$.

Comment. The second lemma can be proved similarly.

V. TOWARDS A NATURAL PROOF OF THE METRIZATION THEOREM FOR SPACE-TIMES

Idea. We would like to describe a degree to which an event a precedes an event b . An ideal way to detect this causal relation is to make sure that some signal emitted at a affects the event b . However, this detection would require that we trace the effects of every event on every other event.

The situation is similar to communications. We would like to be able to set up communication between every two persons a and b . If there are not too many people, we can do it directly. However, when the number of people grows, it becomes impractical to set up direct communication between every two persons, it is better to use retransmission tower and/or satellites. So:

- People who want to communicate send signals to the communication tower.
- At this tower, the signals get amplified, re-routed, and sent to the recipients.

Similarly, instead of sending a signal from each event to every other event, it is reasonable to make a more practical arrangement, in which:

- every event sends signals to special transmission events x_1, x_2, \dots , and then
- each transmitting event broadcasts its own signals to everyone else.

In this arrangement, we can detect that $a \prec b$ if there is a transmission event x_i for which $a \prec x_i$ and $x_i \prec b$.

For this arrangement to detect all possible causality relations, it is necessary to make sure that for every $a \prec b$, there is a transmitting event $x_i \in (a, b)$. Since the intervals (a, b) form a basis of the topology, this means that there should be a point x_i in every open set – i.e., that the sequence x_i should be everywhere dense. Topological spaces in which such a sequence exist are called separable – and our space-time is indeed assumed to be separable. Thus, we can take

the sequence x_i assumed in the separability assumption, and say that $a \prec b$ if and only if $\exists i ((a \prec x_i) \& (x_i \prec b))$.

Idea made slightly more realistic. Similarly to the proof of the lemma, we can make this statement more realistic by reformulating it as

$$\exists i ((i \text{ is not too large}) \& (a \prec x_i) \& (x_i \prec b)).$$

Resulting formula. We already decided and/or derived that:

- the degree to which i is not too large is 2^{-i} ;
- the degree to which $a \prec x_i$ is $g_{x_i}(a)$;
- the degree to which $x_i \prec b$ is $f_{x_i}(b)$,
- “and” corresponds to the product, and “or” (and quantifier $\exists i$) to the sum.

Thus, we arrive at the following formula:

$$\tau(a, b) = \sum_{i=1}^{\infty} 2^{-i} \cdot g_{x_i}(a) \cdot f_{x_i}(b).$$

Resulting proof. It is now relatively easy to prove that this expression is indeed the desired one, i.e., that it satisfies both properties of the kinematic metric.

Proof of the first property of the kinematic metric. The first property is that $\tau(a, b) > 0$ if and only if $a \prec b$. Indeed, if $a \prec b$, then, due to separability, there exists an i for which $a \prec x_i$ and $x_i \prec b$. Due to the lemmas, we thus have $g_{x_i}(a) > 0$ and $f_{x_i}(b) > 0$, hence

$$2^{-i} \cdot g_{x_i}(a) \cdot f_{x_i}(b) > 0$$

and $\tau(a, b) > 0$.

Vice versa, if the sum $\tau(a, b)$ of non-negative numbers is positive, this means that one of the terms in this sum is positive, i.e., there exists an i for which

$$2^{-i} \cdot g_{x_i}(a) \cdot f_{x_i}(b) > 0$$

and thus, $g_{x_i}(a) > 0$ and $f_{x_i}(b) > 0$. Due to the lemmas, this means that $a \prec x_i$ and $x_i \prec b$. So, by transitivity, we get $a \prec b$.

Proof of the second property of the kinematic metric.

The second property is the “anti-triangle inequality”: that if $a \prec b \prec c$, then $\tau(a, c) \geq \tau(a, b) + \tau(b, c)$. To prove this inequality, we will show that for every i , a similar inequality holds for the i -th term in the sum that defines τ :

$$g_{x_i}(a) \cdot f_{x_i}(c) \geq g_{x_i}(a) \cdot f_{x_i}(b) + g_{x_i}(b) \cdot f_{x_i}(c). \quad (1)$$

By the lemmas, $g_{x_i}(a) > 0$ if and only if $a \prec x_i$ and $f_{x_i}(b) > 0$ if and only if $x_i \preceq b$. The product $g_{x_i}(a) \cdot f_{x_i}(b)$ of two non-negative numbers is positive if and only if both numbers are positive, i.e., if $a \prec x_i$ and $x_i \preceq b$. In interval terms, this means that $g_{x_i}(a) \cdot f_{x_i}(b) > 0$ if and only if $x_i \in (a, b)$.

Thus, it is natural to prove the desired inequality by considering all possible locations of x_i with respect to the intervals (a, b) , (b, c) , and (a, c) .

Case when $x_i \notin (a, b)$ and $x_i \notin (b, c)$. In this case, we have $g_{x_i}(a) \cdot f_{x_i}(b) + g_{x_i}(b) \cdot f_{x_i}(c) = 0$. Since the left-hand side $g_{x_i}(a) \cdot f_{x_i}(c)$ of the formula (1) is always non-negative, the inequality is satisfied.

Case when $x_i \in (a, b)$. In this case, since $x_i \prec b$, we cannot have $b \prec x_i$, so $x_i \notin (b, c)$, and $g_{x_i}(b) \cdot f_{x_i}(c) = 0$. Thus, the desired inequality takes a simplified form

$$g_{x_i}(a) \cdot f_{x_i}(c) \geq g_{x_i}(a) \cdot f_{x_i}(b).$$

This simplified inequality follows from the fact that $b \prec c$ and f_{x_i} is an \succeq -monotonic function, so $f_{x_i}(c) \geq f_{x_i}(b)$.

Remaining case – when $x_i \in (b, c)$. In this case, since $b \prec x_i$, we cannot have $x_i \prec b$, so $x_i \notin (a, b)$, and

$$g_{x_i}(a) \cdot f_{x_i}(b) = 0.$$

Thus, the desired inequality takes a simplified form

$$g_{x_i}(a) \cdot f_{x_i}(c) \geq g_{x_i}(b) \cdot f_{x_i}(c).$$

This simplified inequality follows from the fact that $b \prec c$ and g_{x_i} is an \succeq -decreasing function, so $g_{x_i}(a) \geq g_{x_i}(b)$.

The theorem is proven.

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