

Uncertainty in Partially Ordered Sets as a Natural Generalization of Intervals: Negative Information Is Sufficient, Positive Is Not

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Abstract

In many real-life applications, we have an ordered set: a set of all space-time events, a set of all alternatives, a set of all degrees of confidence. In practice, we usually only have a partial information about an element x of this set. This partial information includes positive knowledge: that $a \leq x$ or $x \leq a$ for some known a , and negative knowledge: that $a \not\leq x$ or $x \not\leq a$ for the known a . In the case of a total order, the set of all elements satisfying this partial information is an interval. We show that in the general case of a partial order, the corresponding analogue of an interval is a convex set. We also show that in general, to describe partial knowledge, it is sufficient to have only negative information about x but it is not sufficient to have only positive information.

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1 Formulation of the Problem

Ordered sets are practically important. In many real-life situations, we have a natural order relation. For example:

- causality relation $a \leq b$, meaning that an event a can influence an event b , is a natural order relation on the set of all space-time events;
- a preference relation $a \leq b$, meaning that a user prefers an alternative b to the alternative a ;
- a relation $a \leq b$ between different terms describing the user's degree of confidence, meaning that the degree b describes more confidence than the degree a .

In all these cases, we have a set X – of events, of alternatives, of degrees of confidence – with an order relation \leq .

For each ordered set (X, \leq) , we will use the usual notations $a \geq b \stackrel{\text{def}}{=} b \leq a$, $a < b \stackrel{\text{def}}{=} (a \leq b \ \& \ a \neq b)$, and $a > b \stackrel{\text{def}}{=} b < a$.

How to describe uncertainty in an ordered set. In the ideal case of a *complete* knowledge, we know the exact element $x \in X$. For example, we may know the event, we may know the alternative, or we may know the degree of confidence.

However, frequently, we only have a *partial* information about an element $x \in X$. This partial information comes in the form of a known relation between this (unknown) element and some known elements x_1, \dots, x_n .

For example, about an unknown event e , we know

- that this event e was influenced by events e_1, \dots, e_m ,
- that this event e , in turn, influenced some other events $e'_1, \dots, e'_{m'}$;
- that this event e was *not* influenced by events $e''_1, \dots, e''_{m''}$;
- that events $e'''_1, \dots, e'''_{n'''}$ did not influence this event.

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The first two conditions describe *positive* information about the event e , the last two conditions describe *negative* information about the event e . We want to characterize the set of all the events which satisfy these conditions – both positive and negative.

Let us describe this situation in the general terms.

Definition 1. Let (X, \leq) be an ordered set.

- By a partial knowledge, we mean a tuple $K = (P^-, P^+, N^+, N^-)$ of subsets of the set X .
- We say that an element $x \in X$ is consistent with this partial knowledge if the following four conditions are satisfied:
 - for every $p \in P^-$, we have $p \leq x$;
 - for every $p \in P^+$, we have $x \leq p$;
 - for every $n \in N^-$, we have $n \not\leq x$;
 - for every $n \in N^+$, we have $x \not\leq n$.
- The set of all elements which are consistent with the partial knowledge K will be called an uncertainty set corresponding to K – and denoted by $U(K)$.

Our objectives are:

- to describe sets which can be represented in the form $U(K)$ for some partial knowledge K ; and
- if possible, to find classes of partial knowledges K which are sufficient to describe arbitrary sets $U(K)$.

2 Definitions and the Main Results

Simplest case: total order, finite sets P^\pm and N^\pm . Let us start with the simplest case when the order \leq is a total order (i.e., for every a and b either $a \leq b$ or $b \leq a$), and all four sets P^- , P^+ , N^+ , and N^- forming a partial knowledge are finite.

The finiteness requirement is natural since in practice, at any given moment of time, we only have finite amount of information.

Definition 2. We say that a partial knowledge $K = (P^-, P^+, N^+, N^-)$ is finite if all four sets P^- , P^+ , N^+ , and N^- are finite.

Uncertainty sets corresponding to finite partial knowledge K can be easily described as *intervals*:

Definition 3. By an interval, we mean one of the following sets:

$$\begin{aligned}
 [a, b] &= \{x : a \leq x \leq b\}; & (a, b] &= \{x : a < x \leq b\}; \\
 [a, b) &= \{x : a \leq x < b\}; & (a, b) &= \{x : a < x < b\}; \\
 [a, +\infty) &= \{x : a \leq x\}; & (a, +\infty) &= \{x : a < x\}; \\
 (-\infty, b] &= \{x : x \leq b\}; & (-\infty, b) &= \{x : x < b\}; \\
 & & (-\infty, +\infty) &= X.
 \end{aligned}$$

Proposition 1. Let (X, \leq) be a totally ordered set. Then:

- For every finite partial knowledge K , the uncertainty set $U(K)$ is an interval.
- Every interval I can be represented as $U(K)$ for an appropriate partial knowledge K , namely for a knowledge K formed by at most two one-points sets.

Comment. For readers' convenience, all the proofs are placed at the end of this paper.

Case of arbitrary (not necessarily finite) sets P^\pm and N^\pm . For totally ordered sets, it is also possible to characterize sets which can be represented as uncertainty sets $U(K)$ for some (not necessarily finite) partial knowledge K .

It turns out that the same characterization holds for an arbitrary (not necessarily totally ordered) ordered set (X, \leq) .

Definition 4. A subset $S \subseteq X$ of an ordered set (X, \leq) is called convex if it satisfies the following property:

$$\text{if } s \in S, s' \in S, \text{ and } s \leq x \leq s', \text{ then } x \in S.$$

Proposition 2. Let (X, \leq) be an ordered set. Then:

- For every partial knowledge K , the uncertainty set $U(K)$ is convex.
- Every convex set $S \subseteq X$ can be represented as an uncertainty set $U(K)$ for an appropriate partial knowledge K .

Comment. Thus, convex sets are natural generalizations of intervals – to the case when the partial knowledge is not necessarily finite and the order is not necessarily total.

Positive and negative knowledge. A general partial knowledge includes both *positive* knowledge P^- and P^+ and *negative* knowledge N^- and N^+ . When is each sufficient?

Definition 5. We say that partial knowledge $K = (P^-, P^+, N^-, N^+)$ is:

- positive if $N^- = N^+ = \emptyset$;
- negative if $P^- = P^+ = \emptyset$.

In general, positive knowledge is not sufficient, but negative knowledge is sufficient.

Proposition 3. There exists an ordered set (X, \leq) and a convex set $S \subseteq X$ that cannot be represented as $U(K)$ for any positive partial knowledge K .

Proposition 4. Let (X, \leq) be an ordered set, and let S be a convex subset of X . Then there exists a negative partial knowledge K for which $S = U(K)$.

3 Proofs

Proof of Proposition 1. Let us first prove that for every finite partial knowledge K , the uncertainty set $U(K)$ is an interval. For a totally ordered set, conditions $p_1 \leq x, \dots, p_i \leq x$ corresponding to a set P^- are equivalent to a single condition $p^- \leq x$, where $p^- \stackrel{\text{def}}{=} \max(p_1, \dots, p_i)$.

Similarly, conditions $x \leq q_1, \dots, x \leq q_j$ corresponding to the set P^+ are equivalent to a single condition $x \leq p^+$, where $p^+ \stackrel{\text{def}}{=} \min(q_1, \dots, q_j)$.

Since \leq is a total order, each condition $n_k \not\leq x$ corresponding to the set N^- is equivalent to $x < n_k$. Thus, the conditions $n_1 \not\leq x, \dots, n_l \not\leq x$ corresponding to the set N^- are equivalent to $x < n_1, \dots, x < n_l$, i.e., to $x < n^-$, where $n^- \stackrel{\text{def}}{=} \min(n_1, \dots, n_l)$.

Similarly, each condition $x \not\leq m_k$ corresponding to the set N^+ is equivalent to $m_k < x$. Thus, the conditions $x \not\leq m_1, \dots, x \not\leq m_a$ corresponding to the set N^+ are equivalent to $m_1 < x, \dots, m_a < x$, i.e., to $n^+ < x$, where $n^+ \stackrel{\text{def}}{=} \max(m_1, \dots, m_a)$.

Thus, all the conditions are equivalent to (at most) four inequalities: $p^- \leq x$, $x \leq p^+$, $x < n^-$, and $n^+ < x$ (at most four, since some of the four sets forming the partial knowledge may be empty, in which case there is no inequality to constrain the value x).

The conditions $p^- \leq x$ and $n^+ < x$ are equivalent to a single condition:

- if $p^- \leq n^+$, then these two conditions are equivalent to a single condition $n^+ < x$;
- if $p^- > n^+$, then these two conditions are equivalent to a single condition $p^- \leq x$.

Similarly, the conditions $x \leq p^+$ and $x < n^-$ are equivalent to a single condition:

- if $p^+ \geq n^-$, then these two conditions are equivalent to a single condition $x < n^-$;
- if $p^+ < n^-$, then these two conditions are equivalent to a single condition $x \leq p^+$.

So, the set of (at most) four inequalities are equivalent to (at most) two, and it is easy to check that the set of all the elements x satisfying these two inequalities is an interval.

Vice versa, every interval I is described by two conditions of the type $a \leq x$, $x \leq b$, $a < x$, and $x < b$. Each condition $a < x$ is equivalent to $x \not\leq a$, and each condition $x < b$ is equivalent to $b \not\leq x$. Thus, each interval I can indeed be described as an uncertainty set $U(K)$ for some partial knowledge that consists of at most two one-point sets.

Proof of Proposition 2. Let us first prove that for every partial knowledge K , the uncertainty set $U(K)$ is convex. In other words, we want to prove that if $s \in U(K)$, $s' \in U(K)$, and $s \leq x \leq s'$, then $x \in U(K)$. The fact that $s \in U(K)$ means that the element s satisfies all the necessary conditions of the type $a \leq s$, $s \leq b$, $a \not\leq s$, and $s \not\leq b$. Thus, to prove that $x \in U(K)$, it is sufficient to prove that the element x also satisfies each of these conditions.

We will prove that for each of the above four conditions, if s and s' satisfies the corresponding condition, then the intermediate element x also satisfies this same condition.

- First, we consider the case when $a \leq s$ and $a \leq s'$. Since $a \leq s$, then from $s \leq x$ and transitivity we conclude that $a \leq x$.
- Next, we consider the case when $s \leq b$ and $s' \leq b$. Since $s' \leq b$, then from $x \leq s'$ and transitivity, we conclude that $x \leq b$.
- Third, we consider the case when $a \not\leq s$ and $a \not\leq s'$. We want to prove that in this case, $a \not\leq x$. We can prove this by contradiction. Indeed, if $a \leq x$, then from $x \leq s'$ and transitivity, we would conclude that $a \leq s'$ – and we know that $a \not\leq s'$. Thus, $a \not\leq x$.
- Finally, we consider the case when $s \not\leq b$ and $s' \not\leq b$. We want to prove that in this case, $x \not\leq b$. We can prove this by contradiction. Indeed, if $x \leq b$, then from $s \leq x$ and transitivity, we would conclude that $s \leq b$ – and we know that $s \not\leq b$. Thus, $x \not\leq b$.

Let us now prove that every convex set S can be represented as an uncertainty set $U(K)$ for some partial knowledge K . Indeed, let us show that $S = U(K)$, where $P^- = P^+ = \emptyset$, and

$$N^- = \{x : x \not\leq s \text{ for all } s \in S\}, \quad N^+ = \{x : s \not\leq x \text{ for all } s \in S\}.$$

By definition of the sets N^- and N^+ , every element $s \in S$ satisfies

- the condition $x \not\leq s$ for all $x \in N^-$ and
- the condition $s \not\leq x$ for all $x \in N^+$.

So, every element $s \in S$ belongs to the corresponding uncertainty set $U(K)$: $S \subseteq U(K)$.

To complete our proof, we must also show that, vice versa, every element u of the uncertainty set $U(K)$ belongs to S . By definition, $u \in U(K)$ means that:

- $n \not\leq u$ for all $n \in N^-$, and

- $u \not\leq n$ for all $n \in N^+$.

If we have $s \leq u \leq s'$ for some $s \in S$ and $s' \in S$, then, by convexity, $u \in S$. Thus, to prove our result, it is sufficient to prove the following two statements:

- that there exists an element $s \in S$ for which $s \leq u$, and
- that there exists an element $s' \in S$ for which $u \leq s'$.

We will prove both statements by contradiction.

Let us start with the first statement. Assume that this statement is not true, i.e., that $s \not\leq u$ for all $s \in S$. By definition of the set N^+ , this means that $u \in N^+$. Since $u \leq u$, we conclude that $u \leq n$ for some $n \in N^+$ – namely, for $n = u$. This contradicts to the fact that $u \not\leq n$ for all $n \in N^+$.

Similarly, we can prove the second statement. Assume that this statement is not true, i.e., that $u \not\leq s$ for all $s \in S$. By definition of the set N^- , this means that $u \in N^-$. Since $u \leq u$, we conclude that $n \leq u$ for some $n \in N^-$ – namely, for $n = u$. This contradicts to the fact that $n \not\leq u$ for all $n \in N^-$.

Both statements are proven, thus $u \in S$. The proposition is proven.

Proof of Proposition 3. As the first example let us take, as the ordered set, a real line R with a natural order. Let us show that the convex set $S = (0, +\infty)$ of all positive numbers cannot be represented as an uncertainty set $U(K)$ for any positive partial knowledge K .

Indeed, let us assume that $S = U(K)$ for some positive partial knowledge $K = (P^-, P^+, \emptyset, \emptyset)$. By definition of the set P^+ , every element $p \in P^+$ of this set must be larger than or equal to every element of S . Since no number is larger than every positive real number, we thus conclude that there are no such elements p , i.e., that $P^+ = \emptyset$.

By definition of the set P^- , every element $p \in P^-$ of this set must be larger than or equal to every positive real number. Thus, p cannot be positive, it has to be non-positive.

Now, the value 0 is larger than or equal to every non-positive number, thus it is larger than or equal to every element from the set P^- and so, is in the uncertainty set $U(K)$. However, $0 \notin S$. So, $S \neq U(K)$.

In this example, the two sets are “almost” equal, in the sense that for $P^- = \{x : x \leq 0\}$ and $P^+ = \emptyset$, the corresponding uncertainty set is $[0, +\infty)$ is the closure of the original convex set $S = (0, +\infty)$. Let us give another example where the difference is more drastic.

Let us consider a 2-D analogue of Minkowski space-time R^2 , with the causality-related order

$$(t, x) \leq (t', x') \Leftrightarrow c \cdot (t' - t) \geq |x' - x|,$$

where c is a speed of light. The meaning of this relation is that for an event (t, x) to influence the event (t', x') , it must be possible to get to x' from x in time $t' - t$ with a speed $\frac{|x - x'|}{t' - t}$ not exceeding the speed of light.

For this order, $(t, x) \leq (t', x')$ implies $t \leq t'$.

Let us show that a convex set $S = \{(t, x) : c \cdot t = x\}$ cannot be represented as an uncertainty set $U(K)$ for any positive partial knowledge. Indeed, by definition of the set P^+ , every element $p = (t_p, x_p) \in P^+$ of this set must be causally following every element of S . Since the set S includes events with arbitrarily values of time t , we would thus conclude that $t_p \geq t$ for all real numbers t . Such a value does not exist, so $P^+ = \emptyset$.

Similarly, by definition of the set P^- , every element $p = (t_p, x_p) \in P^-$ of this set must be causally preceding every element of S . Since the set S includes events with arbitrarily values of time t , we would thus conclude that $t_p \leq t$ for all real numbers t . Such a value does not exist, so $P^- = \emptyset$.

Since both P^+ and P^- are empty sets, the corresponding uncertainty set $U(K)$ coincides with the entire space X and is, thus, different from the set $S \neq X$.

Proof of Proposition 4. This statement was, in effect, proven, when we proven Proposition 2.

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