How to Combine Probabilistic and Possibilistic (Expert) Knowledge: Uniqueness of Reconstruction in Yager's (Product) Approach

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Abstract

Often, about the same real-life system, we have both measurement-related probabilistic information expressed by a probability measure $P(S)$ and expert-related possibilistic information expressed by a possibility measure $M(S)$. To get the most adequate idea about the system, we must combine these two pieces of information. For this combination, R. Yager – borrowing an idea from fuzzy logic – proposed to use the simple product t-norm, i.e., to consider a set function $f(S) = P(S) \cdot M(S)$. A natural question is: can we uniquely reconstruct the two parts of knowledge from this function $f(S)$? In this paper, we prove that while in the discrete case, the reconstruction is often not unique, in the continuous case, we can always uniquely reconstruct both components $P(S)$ and $M(S)$ from the combined function $f(S)$. In this sense, Yager's combination is indeed an adequate way to combine the two parts of knowledge.
Keywords: probability measure, possibility measure, Yager's (product) combination, uniqueness of reconstruction

1. Need to combine probabilistic and possibilistic knowledge. One of the main objectives of science is to learn more about the world. One of the main objectives of engineering is to be able to change the world to satisfy certain objectives – build a house, build a road, a computer, etc. In order to make sure that this change is successful, we also need to know the current state of the world.

Most information about the world comes from measurements. As a result of the measurements, we learn the values of relevant quantities $x_1, x_2, \ldots$; together, these values $x_1, \ldots, x_n$ form the description $x = (x_1, \ldots, x_n)$ of the state of the world.

Let X be the set of all such states. In view of the above description, it is reasonable to assume that the state S either coincides with the n-dimensional space $\mathbb{R}^n$ or with an open set in this space. In some cases, we know that only finitely values of each of these variables are possible. In such cases, we have a finite set X of all the states.

We rarely know the exact state x: measurements are usually imprecise, and based on the measurement results, we only have partial knowledge about x. Traditionally, in science and engineering, the corresponding uncertainty is described in probabilistic terms; see, e.g., (Rabinovich 2005).

In the discrete case, we assign a probability $p(x) > 0$ to each possible state $x \in X$, so that $\sum\limits_{x \in X} p(x) = 1$. States x for which the probability is 0 ($p(x) = 0$) are impossible, and can thus be deleted from the set X.

In this case, for every set $S \subseteq X$, the probability $P(S)$ that the actual (unknown) state x belongs to this set can be computed as $P(S) = \sum\limits_{x \in S} p(x)$. Such functions P that assign to every (measurable) set S a probability $P(S)$ are called *probability measures*.

In the continuous case, we assign a probability density $\rho(x) > 0$ to each possible state $x \in X$, so that $\int \rho(x)\,dx = 1$. Usually, the probability density $\rho(x)$ continuously depends on x. For every measurable set $S \subseteq X$, the probability $P(S)$ that the actual (unknown) state x belongs to this set can be computed as $P(S) = \int_S \rho(x)\,dx$.

In addition to measurements, we also have expert knowledge. Expert knowledge is often described in terms of possibilities: an expert assign, to each possible state x, a degree $\mu(x) \in (0, 1]$ to which this state is possible; see, e.g., (Klir and Yuan 1995), (Nguyen and Walker 2006). States x for which the degree of possibility is 0 ($\mu(x) = 0$) are impossible, and can thus be deleted from the set X. Usually, a small change in the state does not change this degree much, so it is reasonable to assume that the function $\mu(x)$ is continuous.

If we have two possible states x and y, then the degree with which it is possible that one of these state is the actual state is equal to the largest of the corresponding degrees: $M(\{x, y\}) = \max(\mu(x), \mu(y))$. Similarly, for each set $S \subseteq X$, it is reasonable to say that the degree $M(S)$ with which it is possible that the actual (unknown) state x is in this set S is equal to the largest of all the degrees $\mu(x)$, $x \in S$, i.e., that $M(S) = \sup_{x \in S} \mu(x)$. The resulting functions $M(S)$ are called *possibility measures*; see, e.g., (Klir and Yuan 1995), (Nguyen 2006), and (Nguyen and Walker 2006).

To get the most adequate understanding of the system, we need to combine the measurement-based probabilistic knowledge and the expert-based possibilistic knowledge.

2. Yager's (product) approach to combining probabilistic and possibilistic (product) knowledge. For each set $S \subset X$, we have two values: the probability $P(S) \in [0, 1]$ that the actual state is in this set *and* the possibility $M(S) \in [0, 1]$ that the actual (unknown) value x is in this case. How can we combine these two values? The need to combine the two degrees d and d' from the interval $[0, 1]$ is well analyzed in fuzzy logic (Klir and Yuan 1995), (Nguyen and Walker 2006), where a special class of "and"-operations (t-norms) has been developed to describe operations corresponding to "and". One of the simplest and widely used t-norms is the product $d \cdot d'$. In view of this, in (Yager 2011) and (Yager to appear), it is proposed to use the product to combine the probabilistic and possibilistic measures into a single set function $f(S) = P(S) \cdot M(S)$.

3. Main question: how uniquely can we reconstruct the probabilistic and possibilistic measures from the combination $f(S)$? A natural question is: once we have this combined measure $f(S)$, can we uniquely reconstruct the original probabilistic and possibilistic measures? In other words, if $P(S) \cdot M(S) = P'(S) \cdot M'(S)$, does it follow that $P(S) = P'(S)$ and $M(S) = M'(S)$ for all sets S?

In this paper, we provide an answer to this question – and to several related auxiliary questions.

4. In the discrete case, reconstruction is not unique. Let us give a simple example of such non-uniqueness, for the simple set $X = \{1, 2\}$. Indeed, let

$$p(1) = 0.2, \quad p(2) = 0.8, \quad \mu(1) = 1, \quad \mu(2) = 0.25,$$

then

$$P(\emptyset) = 0, \quad P(\{1\}) = 0.2, \quad P(\{2\}) = 0.8, \quad P(X) = 1,$$

$$M(\emptyset) = 0, \quad M(\{1\}) = 1, \quad M(\{2\}) = 0.25, \quad M(X) = 1.$$

Thus,

$$f(\{1\}) = P(\{1\} \cdot M(\{1\}) = 0.2 \cdot 1 = 0.2,$$

$$f(\{2\}) = P(\{2\} \cdot M(\{2\}) = 0.8 \cdot 0.25 = 0.2,$$

$$f(X) = P(X) \cdot M(X) = 1 \cdot 1 = 1.$$

The resulting combination function

$$f(\{1\}) = 0.2, \quad f(\{2\}) = 0.2, \quad f(X) = 1$$

is symmetric with respect to swapping 1 and 2. Thus, the same combination function will appear if we "swap" 1 and 2 and consider a different pair of measures:

$$p'(1) = 0.8, \quad p'(2) = 0.2, \quad \mu'(1) = 0.25, \quad \mu'(2) = 1.$$

Non-uniqueness is proven.

5. In the continuous case, reconstruction is unique: a proof. Let us prove that in the continuous case, we can uniquely reconstruct both the probability density $\rho(x)$ and the possibility values $\mu(x)$ from the combination function $f(S) = P(S) \cdot M(S)$.

5.1. Detecting when $\mu(x) < \mu(y)$. Let us first show how, based on the function $f(S)$, we can detect, for every two points x and y, whether $\mu(x) < \mu(y)$.

Indeed, if $\mu(x) < \mu(y)$, then, since the function $\mu(x)$ is continuous, for sufficiently small $\varepsilon > 0$ and $\delta > 0$, we have $\mu(x') < \mu'(y)$ for all x' from the ε-vicinity $B_\varepsilon(x) = \{x' : d(x, x') \leq \varepsilon\}$ of the state x and for all y' from the δ-vicinity $B_\delta(y) = \{y; : d(y, y') \leq \delta\}$ of the state y. Thus,

$$M(B_\varepsilon(x) \cup B_\delta(y)) = \max\left(\sup_{x':d(x,x')\leq\varepsilon} \mu(x'), \sup_{y':d(y,y')\leq\delta} \mu(y')\right) =$$

$$\sup_{y':d(y,y')\leq\delta} \mu(y')$$

and thus, in the limit $\delta \to 0$, we get

$$M(B_\varepsilon(x) \cup B_\delta(y)) \to \mu(y).$$

For the probability measure,

$$P(B_\varepsilon(x) \cup B_\delta(y)) = P(B_\varepsilon(x)) + P(B_\delta(y))$$

and so, in the limit $\delta \to 0$, we get

$$P(B_\varepsilon(x) \cup B_\delta(y)) \to P(B_\varepsilon(x)).$$

Thus, in this case,

$$f(B_\varepsilon(x) \cup B_\delta(y)) = P(B_\varepsilon(x) \cup B_\delta(y)) \cdot M(B_\varepsilon(x) \cup B_\delta(y)) \to P(B_\varepsilon(x)) \cdot \mu(y).$$

For the limit set,

$$f(B_\varepsilon(x)) = P(B_\varepsilon(x)) \cdot M(B_\varepsilon(x)) = P(B_\varepsilon(x)) \cdot \sup_{x':d(x,x')\leq\varepsilon} \mu(x').$$

Since

$$\sup_{x':d(x,x')\leq\varepsilon} \mu(x') < \mu(y),$$

we thus conclude that

$$f(B_\varepsilon(x)) < \lim_{\delta\to 0} f(B_\varepsilon(x) \cup B_\delta(y)).$$

On the other hand, if $\mu(x) \geq \mu(y)$, then for

$$M(B_\varepsilon(x) \cup B_\delta(y)) = \max\left(\sup_{x':d(x,x')\leq\varepsilon} \mu(x'), \sup_{y':d(y,y')\leq\delta} \mu(y')\right),$$

when $\delta \to 0$, we get

$$M(B_\varepsilon(x) \cup B_\delta(y)) \to M(B_\varepsilon(x)).$$

Since

$$P(B_\varepsilon(x) \cup B_\delta(y)) \to P(B_\varepsilon(x)),$$

we conclude that

$$f(B_\varepsilon(x) \cup B_\delta(y)) = P(B_\varepsilon(x) \cup B_\delta(y)) \cdot M(B_\varepsilon(x) \cup B_\delta(y)) \to$$

$$P(B_\varepsilon(x)) \cdot M(B_\varepsilon(x)) = f(B_\varepsilon(x)),$$

and thus,

$$\lim_{\delta \to 0} f(B_\varepsilon(x) \cup B_\delta(y)) = f(B_\varepsilon(x)).$$

So, we can indeed detect whether $\mu(x) < \mu(y)$: this inequality occurs if and only if for all sufficiently small $\varepsilon > 0$, we have

$$f(B_\varepsilon(x)) < \lim_{\delta \to 0} f(B_\varepsilon(x) \cup B_\delta(y)).$$

5.2. Determining the ratio $\mu(y)/\mu(x)$. Let $\mu(x) < \mu(y)$. Then, for small ε, asymptotically,

$$P(B_\varepsilon(x)) \sim \rho(x) \cdot V(B_\varepsilon(x)),$$

where $V(B)$ denotes the volume of the set B, and

$$M(B_\varepsilon(x)) \sim \mu(x).$$

Thus,

$$f(B_\varepsilon(x)) = P(B_\varepsilon(x)) \cdot M(B_\varepsilon(x)) \sim (\rho(x) \cdot \mu(x)) \cdot V(B_\varepsilon(x)).$$

On the other hand, we have

$$\lim_{\delta \to 0} P(B_\varepsilon(x) \cup B_\delta(y)) = P(B_\varepsilon(x)) \sim \rho(x) \cdot V(B_\varepsilon(x)),$$

and

$$\lim_{\delta \to 0} M(B_\varepsilon(x) \cup B_\delta(y)) = \mu(y).$$

Thus,

$$\lim_{\delta \to 0} f(B_\varepsilon(x) \cup B_\delta(y)) = \lim_{\delta \to 0} P(B_\varepsilon(x) \cup B_\delta(y)) \cdot M(B_\varepsilon(x) \cup B_\delta(y)) \sim$$

$$(\rho(x) \cdot \mu(y)) \cdot V(B_\varepsilon(x)).$$

Thus,

$$\frac{\lim\limits_{\delta \to 0} f(B_\varepsilon(x) \cup B_\delta(y))}{P(B_\varepsilon(x))} \sim \frac{(\rho(x) \cdot \mu(y)) \cdot V(B_\varepsilon(x))}{(\rho(x) \cdot \mu(x)) \cdot V(B_\varepsilon(x))} = \frac{\mu(y)}{\mu(x)},$$

and thus, we can determine the ratio $\mu(y)/\mu(x)$ as

$$\frac{\mu(y)}{\mu(x)} = \lim_{\varepsilon \to 0} \frac{\lim\limits_{\delta \to 0} f(B_\varepsilon(x) \cup B_\delta(y))}{P(B_\varepsilon(x))}.$$

When $\mu(y) < \mu(x)$, we can similarly determine the ratio $\mu(x)/\mu(y)$ and thus, reconstruct its inverse $\mu(y)/\mu(x)$.

Finally, when $\mu(y) = \mu(x)$, we can detect this case – in which the ratio $\mu(y)/\mu(x)$ is equal to 1 – because this is the case when $\mu(y) \not> \mu(x)$ and $\mu(x) \not> \mu(y)$, and these inequalities we already know how to detect.

5.3. Finishing the proof. Let us assume that the probability density $\rho'(x)$ and possibility function $\mu'(x)$ lead to the same function

$$f(S) = P(S) \cdot M(S) = P'(S) \cdot M'(S).$$

Since the ratio $\mu(y)/\mu(x)$ is uniquely determined from the function $f(S)$, we conclude that

$$\frac{\mu(y)}{\mu(x)} = \frac{\mu'(y)}{\mu'(x)}.$$

Let us pick any value $x_0 \in X$, then

$$\frac{\mu(y)}{\mu(x_0)} = \frac{\mu'(y)}{\mu'(x_0)},$$

and thus,

$$\mu'(y) = C \cdot \mu(y),$$

where $C \stackrel{\text{def}}{=} \dfrac{\mu(x_0)}{\mu'(x_0)}$. Hence, we have

$$M'(X) = \sup_{x \in X} \mu'(x) = \sup_{x \in X} (C \cdot \mu(x)) = C \cdot \sup_{x \in X} \mu(x) = C \cdot M(X).$$

Since both $P(S)$ and $P'(X)$ are probability measures, we have $P(X) = P'(X) = 1$ and thus, $f(X) = P(X) \cdot M(X) = M(X)$ and $f(X) = P'(X) \cdot M'(X) = M'(X)$. So, $M(X) = M'(X)$ and thus, since $M'(X) = C \cdot M(X)$, we get $C = 1$ and $\mu'(x) = \mu(x)$. Thence, the possibility measures coincide, so for every set S, we have $M(S) = M'(S)$.

Now, from

$$f(S) = P(S) \cdot M(S) = P'(S) \cdot M'(S) = P'(S) \cdot M(S),$$

we conclude that $P(S) = P'(S)$, i.e., that the probability measures also coincide. Uniqueness is proven.

6. Conclusion. The fact that in the continuous case, we can uniquely reconstruct both the possibility and possibility parts of the combined measure means that by performing this combination, we do not lose any information. In this sense, this combination operation is adequate.

7. First auxiliary result: if we know the order of possibilities, then there is uniqueness even in the discrete case. In the previous text, we had an example showing that in the discrete case, reconstruction is not unique. It turns out that if we know which possibilities are larger and which are smaller, then we can uniquely reconstruct both probabilities $p(x)$ and possibilities $\mu(x)$ from the product $f(S) = P(S) \cdot M(S)$.

Indeed, if $\mu(x) \le \mu(y)$, then we have

$$f(\{x\}) = p(x) \cdot \mu(x), \quad f(\{y\}) = p(y) \cdot \mu(y), \quad f(\{x, y\}) = (p(x) + p(y)) \cdot \mu(x).$$

Thus, $f(\{x, y\}) - f(\{y\}) = p(x) \cdot \mu(y)$ and thus, we can reconstruct the ratio $\mu(y)/\mu(x)$ as

$$\frac{\mu(y)}{\mu(x)} = \frac{f(\{x, y\}) - f(\{y\})}{f(\{x\})}.$$

Once we know this ratio, we can reconstruct both $p(x)$ and $\mu(x)$ – as we have shown in the continuous case.

Comment. For n elements, there are $n!$ possible orders. So, if we do not know the order, we have at most $n!$ possible pairs of probability and possibility measures that can lead to a given function $f(S)$.

8. What if we combine only possibility measures? What if instead of combining the probability and possibility measures, we similarly combine two possibility measures? Will we still get uniqueness?

In this case, the answer is no, even in the continuous case. Indeed, let us pick any non-constant continuous function $\mu(x)$ and take $f(S) = M(S) \cdot M(S) = (M(S))^2$. One can easily check that for $\mu'(x) = (\mu(x))^{0.5}$, we get $M'(S) = (M(S))^{0.5}$, and that for $\mu''(x) = (\mu(x))^{1.5}$, we get $M''(S) = (M(S))^{1.5}$. Here,

$$M'(S) \cdot M''(S) = (M(S))^{0.5} \cdot (M(S))^{1.5} = (M(S))^2,$$

hence $M(S) \cdot M(S) = M'(S) \cdot M''(S)$, but $M'(S) \ne M(S)$. So, in this case, reconstruction is not unique.

9. What is we combine only probability measures? Let us prove that in this case, we have uniqueness of reconstruction both in the discrete and in the continuous case. In other words, we prove that if for some probability measures $P_i(S)$ and $P'_i(S)$, we have

$$f(S) = P_1(S) \cdot P_2(S) = P'_1(S) \cdot P'_2(S)$$

for all measurable sets S, then:

- either $P_1(S) = P'_1(S)$ and $P_2(S) = P'_2(S)$ for all measurable sets S,
- or $P_1(S) = P'_2(S)$ and $P_2(S) = P'_1(S)$ for all measurable sets S.

Let us start with the discrete case. In this case, for every two elements x and y, we know the values:

$$f(\{x\}) = p_1(x) \cdot p_2(x), \quad f(\{y\}) = p_1(y) \cdot p_2(y),$$

$$f(\{x, y\}) = (p_1(x) + p_1(y)) \cdot (p_2(x) + p_2(y)).$$

Thus,

$$\Delta(x,y) \stackrel{\rm def}{=} f(\{x,y\}) - f(\{x\}) - f(\{y\}) = p_1(x)\cdot p_2(y) + p_2(x)\cdot p_1(y).$$

The first term $p_1(x)\cdot p_2(y)$ can represented as

$$p_1(x)\cdot p_2(y) = (p_1(x)\cdot p_2(x))\cdot \frac{p_2(y)}{p_2(x)},$$

i.e., as $p_1(x)\cdot p_2(y) = f(\{x\})\cdot r$, where we denoted $r \stackrel{\rm def}{=} \dfrac{p_2(y)}{p_2(x)}$. Similarly, the second term $p_2(x)\cdot p_1(y)$ can be represented as $f(\{y\})\cdot r^{-1}$. Thus, the above formula has the form

$$\Delta(x,y) = f(\{x\})\cdot r + f(\{y\})\cdot r^{-1}.$$

The only unknown here is the ratio r. Multiplying both sides of this equation by r and moving all the terms into one side, we get a quadratic equation

$$f(\{x\})\cdot r^2 - \Delta(x,y)\cdot r + f(\{y\}) = 0,$$

whose solutions are

$$r = \frac{\Delta(x,y) \pm \sqrt{\Delta^2(x,y) - 4\cdot f(\{x\})\cdot f(\{y\})}}{2\cdot f(\{x\})}.$$

Here,

$$\Delta^2(x,y) - 4\cdot f(\{x\})\cdot f(\{y\}) =$$
$$(p_1(x)\cdot p_2(y) + p_2(x)\cdot p_1(y))^2 - 4\cdot p_1(x)\cdot p_2(x)\cdot p_1(y)\cdot p_2(y) =$$
$$(p_1(x)\cdot p_2(y) - p_2(x)\cdot p_1(y))^2,$$

so

$$r = \frac{p_1(x)\cdot p_2(y) + p_2(x)\cdot p_1(y) \pm (p_1(x)\cdot p_2(y) - p_2(x)\cdot p_1(y))}{2\cdot p_1(x)\cdot p_2(x)}.$$

For the plus sign, we get

$$r = \frac{2\cdot p_1(x)\cdot p_2(y)}{2\cdot p_1(x)\cdot p_2(x)} = \frac{p_2(y)}{p_2(x)}.$$

For the minus sign, we get

$$r = \frac{2\cdot p_1(y)\cdot p_2(x)}{2\cdot p_1(x)\cdot p_2(x)} = \frac{p_1(y)}{p_1(x)}.$$

If $f(S) = P_1'(S)\cdot P_2'(S)$, then the corresponding ratio $r = \dfrac{p_2'(y)}{p_2'(x)}$ should satisfy the same quadratic equation, so we have

$$\frac{p_2'(y)}{p_2'(x)} = \frac{p_2(y)}{p_2(x)} \text{ or } \frac{p_2'(y)}{p_2'(x)} = \frac{p_1(y)}{p_1(x)}.$$

A similar argument shows that

$$\frac{p_1'(y)}{p_1'(x)} = \frac{p_2(y)}{p_2(x)} \text{ or } \frac{p_1'(y)}{p_1'(x)} = \frac{p_1(y)}{p_1(x)}.$$

Thus, for each x and y, the ratio $\dfrac{p_2'(y)}{p_2'(x)}$ coincides either with the ratio corresponding to the first original probability measure $P_1(S)$ or with the ratio corresponding to the second original probability measure $P_2(S)$.

To complete the proof for the discrete case, we need to show that this correspondence between the new and the original probability measures is the same for all the pairs. Indeed, let us consider arbitrary three elements x, y and z. In this case, we have three pairs (x, y), (y, z), and (x, z). If the correspondence is different from some of these pairs, this means that we should have two pairs in which the ratio $\dfrac{p_2'(y)}{p_2'(x)}$ is equal to one of the two P_i-ratios and one pair in which this ratio is equal to the other ratio. Without losing generality, let us assume that two ratios are equal to the P_2-ratio and one is equal to the P_1-ratio, i.e., specifically, that

$$\frac{p_2'(y)}{p_2'(x)} = \frac{p_2(y)}{p_2(x)}, \quad \frac{p_2'(z)}{p_2'(y)} = \frac{p_2(z)}{p_2(y)}, \quad \frac{p_2'(z)}{p_2'(x)} = \frac{p_1(z)}{p_1(x)} \neq \frac{p_2(z)}{p_2(x)}.$$

In this case, however, multiplying the first two equalities, we get

$$\frac{p_2'(z)}{p_2'(x)} = \frac{p_2(z)}{p_2(x)},$$

which contradicts to the above inequality. This contradiction shows that for every three elements, the correspondence is the same and thus, it is the same for all pairs: since we can go from each pair (x, y) to every other pair (x', y') via two intermediate triples:

- a triple (x, x', y) that contains pairs (x, y) and (x, x') – and that shows that the pairs (x, y) and (x, x') have the same correspondence, and
- a triple (x', x, y') that contains pairs (x, x') and (x', y') – and that shows that the pairs (x, x') and (x', y') also have the same correspondence.

So, in the discrete case, we either have

$$\frac{p_2'(y)}{p_2'(x)} = \frac{p_2(y)}{p_2(x)}$$

for all x and y, or we have

$$\frac{p_2'(y)}{p_2'(x)} = \frac{p_1(y)}{p_1(x)}$$

for all x and y. In the first case, as in the proof of our main result, we can take any $x_0 \in X$ and prove that for some constant C, we have $p_2'(x) = C \cdot p_2(x)$ for all x. Then, from the fact that both $p'_2(x)$ and $p_2(x)$ are probability measures, we conclude that $C = 1$ and thus, that $p'_2(x) = p_2(x)$ for all x.

In the second case, we similarly have $p'_1(x) = p_1(x)$ for all x. Thus,

- either $P'_2(S) = P_2(S)$ for all S,
- or $P'_2(S) = P_1(S)$ for all S.

A similar result is true for $P'_1(S)$. The result is proven for the discrete case.

In the continuous case, we can partition the space X into finitely many small subsets. Then, for the sets formed by taking unions of these subsets, we have the exactly discrete situation, so for these subsets,

- either we have $P'_1 = P_1$ and $P'_2 = P_2$
- or we have $P'_1 = P_2$ and $P'_2 = P_1$.

We can then take smaller and smaller subsets. For every x and i, we have

$$\rho_i(x) = \lim_{S \ni x,\ S \to x} \frac{P_i(S)}{V(S)}, \quad \rho'_i(x) = \lim_{S \ni x,\ S \to x} \frac{P'_i(S)}{V(S)}.$$

Thus, in the first case, in the limit, we will have $\rho'_1(x) = \rho_1(x)$ and $\rho'_2(x) = \rho_2(x)$ for all x and thus, $P'_1(S) = P_1(S)$ and $P'_2(S) = P_2(S)$ for all measurable sets S.

Similarly, in the second case, in the limit, we will have $\rho'_1(x) = \rho_2(x)$ and $\rho'_2(x) = \rho_1(x)$ for all x and thus, $P'_1(S) = P_2(S)$ and $P'_2(S) = P_1(S)$ for all measurable sets S. The uniqueness is proven.

10. Remaining open questions. What if we have a product of three or more probability measures? Will we be able to uniquely reconstruct all of them? What if we have a product of a possibility measure and two or more probability measures? What is we use a different t-norm instead of the product? Will uniqueness and non-uniqueness results still hold?

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