Dynamic Fuzzy Logic Leads to More Adequate “And” and “Or” Operations

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Abstract

In the traditional (static) fuzzy logic approach, we select an “and”-operation (t-norm) and an “or”-operation (t-conorm). The result of applying these selected operations may be somewhat different from the actual expert’s degrees of belief in the corresponding logical combinations $A \,\&\, B$ and $A \vee B$ of the original statements – since these degrees depend not only on the expert’s degrees of belief in statements A and B, but also in the extent to which the statements A and B are dependent. We show that *dynamic* fuzzy logic enables us to automatically take this dependence into account – and thus, leads to more adequate “and”- and “or”-operations.

1 Formulation of the Problem

“And” and “or” operations in fuzzy logic. In many practical applications, expert rules are formulated by using imprecise (“fuzzy”) words from natural language, like “small”, “medium size”, or “large”. For example, medical recommendations about a tumor depend on whether the tumor is small, medium size, or large. Similarly, expert advise on how to avoid collision with a car in front depend on whether the distance to this car is small, medium, or large. To describe such words, Lotfi Zadeh proposed *fuzzy logic*. In fuzzy logic, to describe each imprecise (“fuzzy”) property P like “small”, we assign, to each possible value x of the corresponding quantity (e.g., the size of a tumor), the degree $\mu_P(x) \in [0, 1]$ to which experts believe that this value satisfies the given property; see, e.g., [3, 5].

This number can come, e.g., as a proportion of the experts who believe that x satisfies the given property, or as a subjective probability.

As a result, for each specific object of size x, we have a degree $\mu_P(x)$ to which this object satisfies the given property and to which, thus, the corresponding expert rule is applicable.

Often, an expert rule contains several conditions: e.g., we can say that if an obstacle is close *and* the car is going fast, then we need to break fast; if a skin tumor is large *or* bleeding *or* has irregular shape, then we need to operate on it. From the description of the corresponding terms, we know the expert’s degrees of confidence in the corresponding component statements: e.g., that the obstacle is close and that the car is going fast. To find the degree to which the rule as a whole is applicable, we need to combine there degrees into a single degree that describes to what extent the corresponding "and" and "or" statements are satisfies. An algorithm $f_\&(a,b)$ that transforms degree of confidence a and b in statements A and B into the degree of confidence in the composite statement $A\,\&\,B$ is called an *"and"-operation* or a *t-norm*. Similarly, an algorithm $f_\vee(a,b)$ that transforms degree of confidence a and b in statements A and B into the degree of confidence in the composite statement $A \vee B$ is called an *"or"-operation* or a *t-conorm*.

Variety of t-norms and t-conorms. In fuzzy logic, there are numerous t-norms and t-conorms. Which one to apply depends on the relation between the statements A and B. This dependence can be illustrated in the probabilistic approaches, when degree a represents the probability that A is true (or the probability that a randomly selected expert considers A to be true).

If A and B are independent, then the probability $f_\&(a,b)$ of $A\,\&\,B$ is equal to the product $a \cdot b = P(A) \cdot P(B)$ of the corresponding probabilities. In this case, the most adequate t-norm is a product $f_\&(a,b) = a \cdot b$.

Of the other hand, if we know that A and B are strongly correlated, then a t-norm $f_\&(a,b) = \min(a,b)$ which leads to $P(A\,\&\,B) = P(A) = P(B)$ if $A = B$ is more adequate.

The problem is that in many cases, we do not know whether A and B are correlated or not. In such cases, we select *some* t-norm. The selected t-norm may not necessarily coincide with the ideal one; hence, the resulting recommendations may not be always adequate. This "truth-functionality", the fact that the degree of confidence in $A\,\&\,B$ depends only on the degrees of confidence in A and B – without fully adequately taking into account the possibility of different correlations – if often cited as one of the main limitations of fuzzy techniques.

Dynamic fuzzy logic. The traditional fuzzy logic assumes that the expert's degrees of confidence do not change. In reality, the expert's opinions often change with time. Thus, to get a more adequate description of the expert opinions and rules, it is necessary to take these changes into account, i.e., to take into account that the expert's degree of confidence in each statement A changes with time. In other words, to describe the expert's opinion about a statement A, instead of a single value $a \in [0,1]$, we need to use a function $a(t)$ that describes how this degree changes with time t. Such *dynamic fuzzy logic* was proposed in [2, 6, 7, 8].

What we do in this paper. In this paper, we show that, if we take this dynamics into consideration, then we can get a more adequate description of "and" and "or" operations, a description in which it is possible to distinguish between the cases when the statements are independent and when they are strongly dependent.

This possibility will be illustrated on the example when the fuzzy degrees have a probabilistic meaning.

2 Main Idea

Relation between correlation and the probabilities $P(A \,\&\, B)$ and $P(A \vee B)$: reminder. In statistics, the most frequent way to describe correlation between two random variables x and y is to use the correlation coefficient

$$\rho = \frac{E[x \cdot y] - E[x] \cdot E[y]}{\sqrt{V[x] \cdot V[y]}},$$

where $E[x]$ denote the mean (expected value) of the variable x and the variance $V[x]$ is defined as

$$V[x] \stackrel{\text{def}}{=} E[(x - E[x])^2] = E[x^2] - (E[x])^2;$$

see, e.g., [9].

A statement A which is true with probability a and false with the remaining probability $1 - a$ can be viewed as a random variable that takes the value 1 (= "true") with probability a and 0 (= "false") with probability $1 - a$. For this variable,

$$E[A] = 1 \cdot a + 0 \cdot (1 - a) = a$$

and similarly, $E[B] = b$. Similarly, $E[A \,\&\, B] = P(A \,\&\, B)$.

Here, $A = 0$ or $A = 1$, hence $A^2 = A$, $E[A^2] = E[A]$ and thus, $V[A] = E[A^2] - (E[A])^2 = a - a^2 = a \cdot (1 - a)$. Similarly, we can conclude that $V[B] = b \cdot (1 - b)$.

For true and false statements, "and" is simply a product, so $A \,\&\, B = A \cdot B$ and thus, $E[A \,\&\, B] = P(A \,\&\, B) = E[A \cdot B]$. Thus, the above formula for the correlation takes the following form:

$$\rho = \frac{P(A \,\&\, B) - a \cdot b}{\sqrt{a \cdot (1 - a) \cdot b \cdot (1 - b)}}.$$

Once we know the probabilities $P(A) = a$ and $P(B) = b$ and the correlation coefficient, we can uniquely reconstruct the probabilities $P(A \,\&\, B)$ and $P(A \vee B)$. From the above formula, we can conclude that

$$P(A \,\&\, B) = a \cdot b + \rho \cdot \sqrt{a \cdot (1 - a) \cdot b \cdot (1 - b)}. \qquad (1)$$

The expression for $P(A \vee B)$ can be found if we take into account the known property

$$P(A \,\&\, B) + P(A \vee B) = P(A) + P(B),$$

from which we conclude that

$$P(A \vee B) = P(A) + P(B) - P(A \,\&\, B) = a + b - P(A \,\&\, B),$$

i.e.,

$$P(A \vee B) = a + b - a \cdot b - \rho \cdot \sqrt{a \cdot (1 - a) \cdot b \cdot (1 - b)}. \qquad (2)$$

How do we find the correlation coefficient? In the dynamic case, we not only know the current expert's degrees of confidence a and b in statements A and B, we also know the past degrees $a(t)$ and $b(t)$ which were, in general, different from a and b.

When the statements A and B are strongly correlated, then it is reasonable to expect that the corresponding changes $a(t)$ and $b(t)$ are also correlated. If the statements A and B are independent, then it is reasonable to expect that the changes $a(t)$ and $b(t)$ are also independent. In general, to find the correlation coefficient between A and B, we can use, as random variables, the values $a(t)$ and $b(t)$ corresponding to T known moments of time. Under this idea,

$$E[A] = \frac{1}{T} \cdot \sum_t a(t), \quad E[B] = \frac{1}{T} \cdot \sum_t b(t),$$

$$V[A] = \frac{1}{T} \cdot \sum_t a^2(t) - \left(\frac{1}{T} \cdot \sum_t a(t)\right)^2, \quad V[B] = \frac{1}{T} \cdot \sum_t b^2(t) - \left(\frac{1}{T} \cdot \sum_t b(t)\right)^2,$$

$$E[A \cdot B] = \frac{1}{T} \cdot \sum_t a(t) \cdot b(t),$$

and thus,

$$\rho = \frac{\frac{1}{T} \cdot \sum_t a(t) \cdot b(t) - \left(\frac{1}{T} \cdot \sum_t a(t)\right) \cdot \left(\frac{1}{T} \cdot \sum_t b(t)\right)}{\sqrt{\left(\frac{1}{T} \cdot \sum_t a^2(t) - \left(\frac{1}{T} \cdot \sum_t a(t)\right)^2\right) \cdot \left(\frac{1}{T} \cdot \sum_t b^2(t) - \left(\frac{1}{T} \cdot \sum_t b(t)\right)^2\right)}}.$$

Substituting this value ρ into (1) and (2), we get the desired estimates for $P(A \,\&\, B)$ and $P(A \vee B)$.

Mathematical comment. In producing these estimates, we implicitly assumed that for the desired statistical characteristic (in our case, correlation), averaging over time leads to the same result as averaging over a sample. This property is called *ergodicity*; it is often assumed and/or proved in statistical physics and in statistical data analysis; see, e.g., [1, 10].

Computational comment. In the above formulas, we implicitly assumed that the correlation between different expert estimates does not change in time. In reality, just like the expert degrees change with time, the correlation between these degrees may also change. It is therefore necessary to take this change into account when estimating correlation. One way to do that is to consider the recent values with higher weights, and the past values with lower weights. In other words, to each of T moments of time, we assign a weight $w(t) \geq 0$ such that $\sum_t w(t) = 1$, and then consider the modified formulas

$$E[A] = \sum_t w(t) \cdot a(t), \quad E[B] = \sum_t w(t) \cdot b(t),$$

$$V[A] = \sum_t w(t) \cdot a^2(t) - \left(\sum_t w(t) \cdot a(t)\right)^2,$$

$$V[B] = \sum_t w(t) \cdot b^2(t) - \left(\sum_t w(t) \cdot b(t)\right)^2,$$

$$E[A \cdot B] = \sum_t w(t) \cdot a(t) \cdot b(t).$$

The above case corresponds to $w(t) = \dfrac{1}{T}$. A usual selection of "discount" weights is $w(t) = C \cdot q^t$ for some $q < 1$. In this case, the sum $\sum w(t) = \sum C \cdot q^t$ is the sum of a geometric progression:

$$\sum_{t=1}^{T} C \cdot q^t = C \cdot \sum_{t=1}^{T} q^t = C \cdot \frac{1 - q^{T+1}}{1 - q}.$$

Thus, once q is selected, the value C is determined from the condition that $\sum_t w(t) = 1$, as

$$C = \frac{1 - q}{1 - q^{T+1}}.$$

3 Limitations: Computational Complexity and Non-Associativity

What are the limitations of this approach?

Computational complexity: description. An obvious limitation is that to find the degree of confidence in $A \,\&\, B$ or in $A \vee B$, we now need to perform a large number of computations – instead of simply applying a t-norm or a t-conorm to two numbers.

Computational complexity is unavoidable. This limitation is unavoidable: in the dynamic fuzzy logic, we have more values for representing the expert's degree of confidence in each statement, so processing these degrees takes more computation time.

Non-associativity: description. Another limitation is that, in contrast to the usual (static) fuzzy logic, dynamic logic operations are not necessarily associative, i.e., the estimates for $(A \vee B) \vee C$ and for $A \vee (B \vee C)$ are, in general, different.

Non-associativity is unavoidable. Let us show that this non-associativity is also a limitation not of a specific *method* of extending "and"- and "or"-operations to dynamic fuzzy logic, but a limitation of the very *dynamic character* of these logics.

Let us show that non-associativity occurs even if we restrict ourselves to linear operations. This possibility comes from the fact that one of the most frequently used probability-related fuzzy "or"-operations $f_\vee(a,b) = a + b - a \cdot b$ is approximately linear for small a and b, and that it is isomorphic to $a + b$ if we appropriately re-scale the values from the interval $[0,1]$ to the set $\mathbb{R}_0^+$ of all non-negative numbers.

Definitions.

- *For every integer t, by a* dynamical fuzzy value *corresponding to time t, we mean a sequence of values $a = \{a_s\}_{s \le t}$, where each value a_s belongs to the set $\mathbb{R}_0^+$.*

- *For every integer t_0 and for each dynamic fuzzy value a, by a* shift *$S_{t_0}(a)$, we mean a sequence $a' = \{a'_s\}_{s \le t+t_0}$ for which $a'_s = a_{s-t_0}$.*

- *By a* aggregation operation, *we mean an operation f that transforms two sequences $a = \{a_s\}_{s \le t}$ and $b = \{b_s\}_{s \le t}$ into a value $c_t \in \mathbb{R}_0^+$.*

- *An operation f is called* shift-invariant *if for every a, b, and t, whenever it transforms a and b into a value c_t, it transforms shifted values $S_{t_0}(a)$ and $S_{t_0}(b)$ into the same value c_{t+t_0}.*

- *We say that an aggregation operation f is* linear *if it is a linear function of all its variables a_s and b_s, i.e.,*

$$c_t = Z_t + \sum_{s \le t} A_{t,s} \cdot a_s + \sum_{s \le t} B_{t,s} \cdot b_s.$$

- *For any aggregation operation f, by the* result *$c = \{c_s\}_{s \le t} = f(a, b)$ of applying this operation to sequences $a = \{a_s\}_{s \le t}$ and $b = \{b_s\}_{s \le t}$ we mean a sequence for which, for every $s \le t$, $c_s = f(\{a_u\}_{u \le s}, \{b_u\}_{u \le s})$.*

- *We say that an operation is* commutative *if $f(a, b) = f(b, a)$ for all a and b, and* associative *if $f(f(a, b), c) = f(a, f(b, c))$.*

Proposition. *If $c = f(a, b)$ is a shift-invariant linear commutative and associative operation, then the value c_t depends only on a_t and b_t and does not depend on the values a_s and b_s for $s < t$.*

Comment. In other words, any commutative linear operation that takes into account previous fuzzy estimates is not associative.

Proof.

1°. Let us first use the fact that our linear aggregation operation is shift-invariance.

By definition, shift-invariance means that for every two sequences a and b, if

$$c_t = Z_t + \sum_{s \le t} A_{t,s} \cdot a_s + \sum_{s \le t} B_{t,s} \cdot b_s,$$

and we combine the shifted sequences $a' = S_{t_0}(a)$ and $b' = S_{t_0}(b)$:

$$c'_{t+t_0} = Z_{t+t_0} + \sum_{s \le t+t_0} A_{t+t_0,s} \cdot a'_s + \sum_{s \le t} B_{t+t_0,s} \cdot b'_s,$$

then we should get the same result: $c_t = c'_{t+t_0}$. Substituting $a'_s = a_{s-t_0}$ and $b'_s = b_{s-t_0}$ into the formula for c'_{t+t_0}, we conclude that

$$c'_{t+t_0} = Z_{t+t_0} + \sum_{s\le t+t_0} A_{t+t_0,s} \cdot a_{s-t_0} + \sum_{s\le t} B_{t+t_0,s} \cdot b_{s-t_0}.$$

Introducing a new variable $s' \stackrel{\text{def}}{=} s - t_0$ for which $s = s' + t_0$, we get

$$c'_{t+t_0} = Z_{t+t_0} + \sum_{s'\le t} A_{t+t_0,s'+t_0} \cdot a_{s'} + \sum_{s\le t} B_{t+t_0,s'+t_0} \cdot b_{s'}.$$

For the following arguments, it is convenient to rename s' into s, as a result, we conclude that

$$c'_{t+t_0} = Z_{t+t_0} + \sum_{s\le t} A_{t+t_0,s+t_0} \cdot a_s + \sum_{s\le t} B_{t+t_0,s+t_0} \cdot b_s.$$

The fact that c_t and c'_{t+t_0} are equal means that the following equality holds for all possible sequences a and b:

$$Z_t + \sum_{s\le t} A_{t,s} \cdot a_s + \sum_{s\le t} B_{t,s} \cdot b_s =$$

$$Z_{t+t_0} + \sum_{s\le t} A_{t+t_0,s+t_0} \cdot a_s + \sum_{s\le t} B_{t+t_0,s+t_0} \cdot b_s.$$

Two linear functions coincide if and only if all their coefficients coincide. Thus, for every t and t_0, we have $Z_t = Z_{t+t_0}$, $A_{t,s} = A_{t+t_0,s+t_0}$, and $B_{t,s} = B_{t+t_0,s+t_0}$.

1.1°. Let us first use the first equality $Z_t = Z_{t+t_0}$.

For every two values t and t', we can take $t_0 = t' - t$, then $t + t_0 = t'$ hence $Z_t = Z_{t'}$. Thus, every two values Z_t coincide, so the value Z_t does not depend on t. We will denote this common value by Z.

1.2°. From $A_{t,s} = A_{t+t_0,s+t_0}$, by taking $t_0 = -s$, we conclude that $A_{t,s} = A_{t-s,0}$. By denoting $A_t \stackrel{\text{def}}{=} A_{t,0}$, we can describe this as $A_{t,s} = A_{t-s}$.

1.3°. Similarly, we conclude that $B_{t,s} = B_{t-s}$.

Thus, a shift-invariant linear operation has the form

$$c_t = Z + \sum_{s\le t} A_{t-s} \cdot a_s + \sum_{s\le t} B_{t-s} \cdot b_s.$$

2°. Let us now use commutativity.

Commutativity means that the result of applying this operation to a and b is the same as the result of applying it to b and a, i.e., that

$$Z + \sum_{s\le t} A_{t-s} \cdot a_s + \sum_{s\le t} B_{t-s} \cdot b_s = Z + \sum_{s\le t} A_{t-s} \cdot b_s + \sum_{s\le t} B_{t-s} \cdot a_s.$$

Here again, the fact that the two linear functions coincide means that all their coefficients must coincide, i.e., that $A_t = B_t$ for all t. Thus, the above formula for c_t takes the form

$$c_t = Z + \sum_{s \le t} A_{t-s} \cdot (a_s + b_s).$$

3°. Let us now use associativity $f(f(a,b),c) = f(a, f(b,c))$.

Associativity means that the two aggregation expressions $f(f(a,b),c)$ and $f(a, f(b,c))$ coincide.

3.1°. In the expression $f(f(a,b),c)$, we first combine sequences a and b into a new sequence $d = f(a,b)$, and then combine d and c into a new sequence $e = f(d,c)$. For the t-th component of these two new sequences d and e, if we keep track only of the dependence on a_t, b_t, and c_t, we get $d_t = A_0 \cdot (a_t + b_t) + \ldots$ and thus,

$$e_t = A_0 \cdot (d_t + c_t) + \ldots = A_0^2 \cdot (a_t + b_t) + A_0 \cdot c_t + \ldots$$

A similar expression for $f(a, f(b,c))$ takes the form

$$A_0^2 \cdot (b_t + c_t) + A_0 \cdot a_t + \ldots$$

The fact that the two expression coincide means that for all possible values a_t, b_t, and c_t, we have

$$A_0^2 \cdot (a_t + b_t) + A_0 \cdot c_t = A_0^2 \cdot (b_t + c_t) + A_0 \cdot a_t.$$

Since the two linear functions coincide, their coefficients must coincide, i.e., we must have $A_0 = A_0^2$. Thus, we have $A_0 = 0$ or $A_0 = 1$.

3.2°. Let us show that in both cases $A_0 = 0$ and $A_0 = 1$, we have $A_1 = A_2 = \ldots = 0$, i.e., the value c_t depends only on a_t and b_t and does not depend on the previous values a_s and b_s.

In both cases, we will prove it by contradiction. Indeed, let us assume that $A_j \neq 0$ for some $j \ge 1$; let k denote the smallest index $k \ge 0$ for which $A_k \neq 0$.

3.2.1°. When $A_0 = 0$, this means that the aggregation operation $f(a,b)$ leads to

$$d_t = Z + A_k \cdot (a_{t-k} + b_{t-k}) + \ldots$$

and

$$e_t = Z + A_k \cdot (d_{t-k} + c_{t-k}) + \ldots$$

Here,

$$d_{t-k} = Z + A_k \cdot (a_{t-2k} + b_{t-2k}) + \ldots$$

Thus, we have

$$e_t = Z + A_k \cdot Z + A_k^2 \cdot (a_{t-2k} + b_{t-2k}) + A_k \cdot c_{t-k} + \ldots$$

Similarly, the second expression $f(a, f(b,c))$ leads to

$$e_t = Z + A_k \cdot Z + A_k^2 \cdot (b_{t-2k} + c_{t-2k}) + A_k \cdot a_{t-k} + \ldots,$$

thus

$$Z + A_k \cdot Z + A_k^2 \cdot (a_{t-2k} + b_{t-2k}) + A_k \cdot c_{t-k} + \ldots =$$

$$Z + A_k \cdot Z + A_k^2 \cdot (b_{t-2k} + c_{t-2k}) + A_k \cdot a_{t-k} + \ldots$$

The left-hand side of this equality does not depend on the value a_{t-k}, it only depends on the previous values, while the right-hand side explicitly depends on a_{t-k}: this term enters with a coefficient $A_k \neq 0$.

Thus, the equality is indeed impossible.

3.2.2°. When $A_0 = 1$, the aggregation operation $f(a, b)$ leads to

$$d_t = Z + a_t + b_t + A_k \cdot (a_{t-k} + b_{t-k}) + \ldots$$

and

$$e_t = Z + d_t + c_k + A_k \cdot (d_{t-k} + c_{t-k}) + \ldots$$

Here,

$$d_{t-k} = Z + a_{t-k} + b_{t-k} + \ldots$$

Thus, we have

$$e_t = Z + (Z + a_t + b_t + A_k \cdot (a_{t-k} + b_{t-k}) + \ldots) + c_t +$$

$$A_k \cdot ((Z + a_{t-k} + b_{t-k} + \ldots) + c_{t-k}) + \ldots =$$

$$2Z + a_t + b_t + c_t + A_k \cdot (2a_{t-k} + 2b_{t-k} + c_{t-k}) + \ldots$$

Similarly, the second expression $f(a, f(b, c))$ leads to

$$e_t = 2Z + a_t + b_t + c_t + A_k \cdot (2b_{t-k} + 2c_{t-k} + a_{t-k}) + \ldots,$$

thus

$$2Z + a_t + b_t + c_t + A_k \cdot (2a_{t-k} + 2b_{t-k} + c_{t-k}) + \ldots =$$

$$2Z + a_t + b_t + c_t + A_k \cdot (2b_{t-k} + 2c_{t-k} + a_{t-k}) + \ldots$$

Since the two linear functions coincide, all corresponding coefficients must coincide. The left-hand side of this equality contains a_{t-k} with a coefficient $2A_k$, while the right-hand side has this variable with a different coefficient $A_k \neq 2A_k$.

Thus, the equality is impossible in this case as well.

The proposition is proven.

Comment. The fact that not all algebraic properties can be satisfied in the dynamical case is known in other similar situations: e.g., in [4], it is proven that if we formulate natural requirements for a reasonable next step in a bargaining process, then every function satisfying these requirements does not depend on the bargaining pre-history.

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