How to Make Sure that Students Spend Enough Time Studying: Fuzzy-Motivated Optimization Approach to Selecting a Grading Policy

Olga Kosheleva
Department of Teacher Education
University of Texas at El Paso
500 W. University
El Paso, TX 79968, USA
Email: olgak@utep.edu

Karen Villaverde
Department of Computer Science
New Mexico State University
Las Cruces, NM 88003, USA
Email: kvillave@cs.nmsu.edu

Abstract—Students do not always spend enough time studying. How can we encourage them to study more? In this paper, we show that a lot depends on the grading policy. At first glance, the problem of grading may seem straightforward: since our objective is that the students gain the largest amount of knowledge and skills at the end of the class, the grade should describe this amount. We show, however, that it is exactly this seemingly straightforward grading policy that often leads to an unfortunate learning behavior. To improve the students’ learning, it is therefore necessary to use a grading policy which goes beyond the straightforward approach. In this paper, we use fuzzy-motivated intuition to formulate selection of a grading policy as a precise optimization problem, and, in the first approximation, provide a solution to this optimization problem. This solution is in line with what experienced instructors are actually doing when grading the class.

I. WHY STUDENTS DO NOT ALWAYS STUDY WELL

Empirical fact. It is known that not all students spend as much time studying as they should. As a result, their knowledge (as judged by their grades) is not as good as it could be if they studied more.

How to improve the situation. It is desirable to make sure that the students spend more time studying. To be able to do that, we first need to understand the students’ reasoning, to understand why they do not always study enough, and then use this understanding to propose ways to encourage them study more.

What we do in this paper. In this paper, we show that the amount of time that the students spend studying depends on the grading policy. It is therefore desirable to come up with a grading policy under which the students will, at the end of the class, gain the largest possible amount of knowledge.

II. FUZZY-MOTIVATED APPROACH: QUALITATIVE ANALYSIS

Qualitative analysis of the problem. Before we formulate quantitative models, let us start with a qualitative analysis of the problem.

Students usually want to succeed, to gain the maximum amount of knowledge in the class – at least in the classes from their major, classes that are of interest to them. On the other hand, students usually do not want to spend all their time studying, they also want to “have a life”; from this viewpoint, the students want to minimize their study efforts.

In decision making terms, this means that the students solve a multi-objective optimization problem.

Need for a fuzzy approach. Strictly speaking, the two objectives of a student, while reasonable, are inconsistent. From the purely mathematical viewpoint, if a student really wants to gain the maximum amount of knowledge, this student should spend all free time studying.

Vice versa, if a student really wants to minimize the study efforts, the student should not study at all.

What these weird conclusions mean is that the above “maximization” and “minimization” should not be taken literally, in the crisp sense, they should be taken informally, in fuzzy sense.

Fuzzy logic provides a natural description of such “fuzzy optimization”: see, e.g., [2], [3].

Comment. It is worth mentioning that fuzzy-related optimization techniques have been successfully used in education: e.g., in selecting the optimal way of presenting the material; see, e.g., [4]. In this paper, we apply these techniques to the problem of designing the optimal grading policy.

Simplifying assumptions. To describe this formalization, let us agree to gauge the amount $V$ knowledge that a student gains in the class as the portion of what this student can potentially gain. In this description, having learned nothing means $V = 0$ and having learned everything that was taught in this class means $V = 1$.

For simplicity, we will also assume:

- that the grade is proportional to the amount of knowledge, and
- that the effort needed to achieve this grade is proportional to this amount as well.
The second of these assumptions means that the students study in a correct way (and study the correct things), so that their study efforts are not wasted. It also means that we gauge the amount of knowledge not simple the number of learned ideas, skills, and results, but that we also take into account the relative difficulty of these learned ideas, skills, and results – so that a more complex idea is worth more than a simpler one.

Case study: US grading system. In this paper, we will illustrate our ideas on the example of the typical US grading system, but, of course, the same ideas are applicable to any grading system.

In the US, usually, the instructor first computes the numerical grade from 0 to 100, 0 meaning no knowledge at all and 100 means perfect knowledge. Based on this numerical grade, the instructor then computes a letter grade that goes into the final grade. The whole interval [0, 100] is usually divided into 5 sections, each corresponding to a different letter grade. The pass mark is an “average” letter grade C, and the value 1 for V corresponds to the value 1.0: the minimum knowledge needed to pass the class.

For values a little bit smaller than 0.6, the student did not pass the class, so his or her degree of satisfaction with the knowledge acquired in this class is 0: \( \mu_k(0.6 - \varepsilon) = 0 \). By continuity, when \( \varepsilon \) tends to 0, we conclude that \( \mu_k(0.6) = 0 \) – and that \( \mu_k(V) = 0 \) for all \( V \leq 0.6 \).

When a student achieved perfect knowledge \( V = 1 \), the degree of satisfaction with the knowledge acquired in the class is 1.0: \( \mu_k(1.0) = 1 \).

We now need to interpolate this membership function to the whole interval \([0.6, 1.0]\). Since we are performing a qualitative analysis anyway, it is reasonable to take the simplest possible interpolation, i.e., a linear function that takes the value 0 for \( V = 0.6 \) and the value 1 for \( V = 1 \); this approach works very well in many practical applications of fuzzy, e.g., in fuzzy control, where the resulting trapezoid and triangular membership functions lead to good results. In our cases, this interpolation leads to the membership function

\[
\mu_k(V) = \frac{V - 0.6}{1.0 - 0.6} = \frac{V - 0.6}{0.4}.
\]

Membership function for effort. In a similar way, we can propose a membership function \( \mu_e(V) \) for minimizing effort.

- When a student does not study at all, i.e., when \( V = 0 \), the goal of minimizing effort is absolutely fulfilled, i.e., \( \mu_e(0) = 1 \).
- On the other hand, when a student spends all the time studying, i.e., when \( V = 1 \), then the goal of minimizing the effort is not satisfied at all, i.e., \( \mu_e(1) = 0 \).

We can now use linear interpolation to find the values of this membership function \( \mu_e(V) \) for all intermediate values \( V \in (0, 1) \). As a result, we get the membership function

\[
\mu_e(V) = 1 - V.
\]

Combining the two objectives into a single objective function. A student selects a value \( V \) that satisfies both objectives: maximizes knowledge and minimizes effort.

- The degree to which the value \( V \) satisfies the first objective, i.e., to which this value maximizes knowledge, is described by the value \( \mu_k(V) \).
- The degree to which the value \( V \) satisfies the second objective, i.e., to which this value saves the efforts, is described by the value \( \mu_e(V) \).

Thus, in accordance with the general fuzzy techniques, the degree to which the value \( V \) satisfies both student’s objectives is described by the value \( f_{k,e}(\mu_k(V), \mu_e(V)) \), where \( f_{k,e}(a, b) \) is a t-norm (“and”-operation).

Similarly to the our choice of the simplest membership function, it is reasonable to choose the simplest t-norm, i.e., the t-norm \( f_{k,e}(a, b) = \min(a, b) \). For this t-norm, the degree to which a given value \( V \) satisfies both student’s objectives is equal to \( \mu(V) = \min(\mu_k(V), \mu_e(V)) \). The student then selects a value \( V \) for which this degree is the largest possible.

Resulting student’s behavior. For values \( V \leq 0.7 \), we have \( \mu_k(V) = 0 \) and thus, \( \mu(V) = \min(\mu_k(V), \mu_e(V)) = 0 \). One can easily check that:

- The function \( \mu(V) \) first coincides with the first membership function \( \mu_k(V) \) (and thus, increases) while
  \[
  \mu_k(V) \leq \mu_e(V).
  \]
- Then, the function \( \mu(V) \) coincides with the second membership function \( \mu_e(V) \) (and thus, decreases) while
  \[
  \mu_e(V) \leq \mu_k(V).
  \]

So, this function \( \mu(V) = \min(\mu_k(V), \mu_e(V)) \) attains its maximum at a value \( V_m \) for which \( \mu_k(V_m) = \mu_e(V_m) \), i.e., for which

\[
\frac{V_m - 0.6}{0.4} = 1 - V_m.
\]

Multiplying both sides by 0.4, we get \( V_m - 0.6 = 0.4 - 0.4 \cdot V_m \), hence \( 1.4 \cdot V_m = 0.6 + 0.4 = 1 \) and

\[
V_m = \frac{1}{1.4} \approx 0.7.
\]
So, the average grade is close to C, which is exactly what we observe in many classes.

**What if only C is a passing grade.** In some classes, a student needs to get at least a grade of C to pass. For example, in the undergraduate Computer Science program of the University of Texas at El Paso, each class which is a pre-requisite for another required class has to be passed with at least a C grade. As we have mentioned, the C grade corresponds to 0.7. Thus, in this case, \( \mu_k(0.7 - \varepsilon) = 0 \) and \( \mu_k(1.0) = 1.0 \), so linear interpolation leads to

\[
\mu_k(V) = \frac{V - 0.7}{1.0 - 0.7} = \frac{1.0 - 0.7}{0.3}. 
\]

Thus, the function \( \mu(V) = \min(\mu_k(V), \mu_m(V)) \) attains its maximum at a value \( V_m \) for which \( \mu_k(V_m) = \mu_e(V_m) \), i.e., for which

\[
\frac{V_m - 0.7}{0.3} = 1 - V_m. 
\]

Multiplying both sides by 0.3, we get \( V_m - 0.7 = 0.3 - 0.3 \cdot V_m \), hence \( 1.3 \cdot V_m = 0.7 + 0.3 = 1 \) and

\[
V_m = \frac{1}{1.3} \approx 0.78. 
\]

So, in such classes, the average grade is close to B, which is also exactly what we observe in many such classes.

**III. FROM QUALITATIVE TO QUANTITATIVE ANALYSIS: STRAIGHTFORWARD APPROACH**

**Towards a quantitative description.** Let \( v(t) \) denote a rate with which a student studies at moment \( t \), i.e., the amount of new knowledge per unit time acquired in the vicinity of this moment.

In this notation, from moment \( t \) to moment \( t + \Delta t \), a student acquires the amount of knowledge \( v(t) \cdot \Delta t \). Thus, the total amount of knowledge acquired by a moment \( t_0 \) can be described by adding all the amounts acquired at the previous moments of time, i.e., as \( \sum v(t) \cdot \Delta t \). From the mathematical viewpoint, this is an integral sum, and in the limit \( \Delta t \to 0 \), it tends to the integral

\[
\int_0^{t_0} v(t) \, dt. 
\]

**Comment.** In this simplified description, we assume that once a student learned some subject, the student will retain this knowledge, i.e., that there is no forgetting.

**Describing effort.** We have assumed that the effort used is proportional to the amount learned. Let us denote the corresponding proportionality coefficient by \( K \).

Of course, the displeasure caused by studying should be smaller than the pleasure caused by acquiring knowledge, otherwise the student will not study at all. So, we must have \( K < 1 \).

**First idea: straightforward evaluation.** Our objective is that the student learns as much as possible at the end of the class, i.e., during the time \( T \) that a class takes. In other words, our objective is to maximize the amount that the student learned during the class, i.e., the value \( \int_0^T v(t) \, dt \). Therefore, it seems reasonable to make the final grade for this class proportional to this value.

**Comment.** This idea is used, e.g., in Russian universities, where all the midterm exams and quizzes only determine whether a person is allowed to take the final exam, and the grade for the class is determined exclusively by this comprehensive final exam.

**Towards analysis of the straightforward approach.** In the straightforward approach, a student gets, at time \( T \), the grade proportional to the integral \( \int_0^T v(t) \, dt \).

**Discounting of future utilities: general description.** In decision theory, it is usually assumed that expected future events that will occur at moment \( f \) in the future are “discounted” with the factor \( e^{-\alpha f} \), where the parameter \( \alpha \) depends on the individual; see, e.g., [5].

**Discounting: financial explanation.** This discounting is easy to explain for financial gains. Indeed, if we are promised an amount \( A \) at time \( f \) in the future, then we can gain the same amount at this future time if we get now an amount \( a \) and invest it so that it will bring some interest (e.g., place it into the saving account). After \( f \) years, the original amount \( a \) will turn into \((1 + r)^f\), where \( r \) is the yearly interest rate. Thus, to get the amount \( A \) in the future, we need now to select the amount \( a \) for which \( a \cdot (1 + r)^f = A \), i.e., the amount

\[
a = (1 + r)^{-f} \cdot A. 
\]

This is exactly the above discounting, with \( \alpha = \ln(1 + r) \).

**Comment.** In the Appendix, we show that The same formula for discounting can be derived from first principles, without using specifics of financial situations.

**Analysis of the straightforward approach (cont-d).** Because of the discounting, the student’s expected gain at the moment the class starts is equal to

\[
e^{-\alpha T} \cdot \int_0^T v(t) \, dt. 
\]

At each moment of time \( t \), the rate \( v(t) \) with which a student studies leads to this student’s rate of effort spending equal to \( K \cdot v(t) \). Because of the discounting, the effect of all these efforts at the beginning of the class is equal to

\[
K \cdot \int_0^T e^{-\alpha t} \cdot v(t) \, dt. 
\]

At the beginning of the semester, the overall utility of a study plan \( v(t) \) to a student is equal to the sum of the positive
utility caused by knowledge and the negative utility caused by efforts, i.e., to
\[ U = e^{-\alpha \cdot T} \int_0^T v(t) \, dt - K \cdot \int_0^T e^{-\alpha \cdot t} \cdot v(t) \, dt. \]

In general, at a moment \( t_0 \), when the student makes a decision about the learning rate \( v(t_0) \) to select at this moment of time, the part of the utility that depends on the remaining selections \( v(t) \) \( (t \geq t_0) \) takes the form
\[ U(t_0) = e^{-\alpha \cdot (T-t_0)} \cdot \int_{t_0}^T v(t) \, dt - K \cdot \int_{t_0}^T e^{-\alpha \cdot (t-t_0)} \cdot v(t) \, dt. \]

This expression is a linear function of all the values \( v(t) \):
\[ U = \int_{t_0}^T c(t) \cdot v(t) \, dt, \]
where \( c(t) = e^{-\alpha \cdot (T-t_0)} - K \cdot e^{-\alpha \cdot (t-t_0)}. \)

Thus, maximizing this expression over all possible non-negative values \( v(t_0) \) means that we select \( v(t_0) = 0 \) when \( c(t_0) = e^{-\alpha \cdot (T-t_0)} < K \cdot e^{-\alpha \cdot (t-t_0)} \) or, equivalently,
\[ e^{-\alpha \cdot (T-t_0)} < K. \]

**Specifics of the young people decision making: brief reminder.** Many students are young people, and for young people, the discount coefficient \( \alpha \) is often high. This is shown by the fact that they often do not consider possible negative future consequences of their actions and vice versa, they do not value possible future positive effects, they live more in the today and they do not care that much about their future.

**Unexpected negative consequence of the straightforward approach.** As a result of the high value \( \alpha \) typical for young people, we get \( e^{-\alpha \cdot T} \ll K \) (and thus, \( \alpha \cdot T \gg |\ln(K)| \)). Hence, for \( t_0 = 0 \), we get \( c(t_0) < 0 \) and thus, \( v(t_0) = 0 \). This means that at the beginning, students do not study – because they have no incentive to study.

Moreover, the students do not start studying until the moment \( t_s \) at which the coefficient \( c(t_s) \) turns from negative to positive, i.e., at which \( e^{-\alpha \cdot (T-t_s)} = K \) and thus, \( \alpha \cdot (T-t_s) = |\ln(K)| \). Since \( \alpha \cdot T \gg |\ln(K)| \), this means that \( \alpha \cdot T \gg \alpha \cdot (T-t_s) \), i.e., \( T \gg T-t_s \). Thus, the time period \( T-t_s \) during which the students actually study is much smaller than the total period \( T \) during which they could potentially study.

Since a human ability to learn the material in a given time is limited, this means that the students end up not learning as much as they potentially could.

**Possible solution to the problem.** The above problem was caused by the way we assign the grade. So, to resolve the problem, it is therefore desirable to modify the way we assign the grade for the class: assigning the grade simply based on the knowledge at the end of the class often does not lead to good results.

**IV. Grading Policy that Leads to Optimal Learning:** Precise Formulation of the Optimization Problem and Its Solution

**Main idea.** The main problem with the straightforward grading scheme is that the final grade is only determined by the final knowledge. This grade becomes known only a long time after the class starts, and this long delays drastically decreases the effect of this grading on the student behavior.

A way to solve this problem is to decrease the time delay between the student learning the material and the student being tested on this material. To achieve this objective, we must introduce earlier (intermediate) tests.

**Comment.** In the US, such tests are often known – somewhat confusingly – as *midterm exams*. This term originated from the setting in which there is one such exam, given exactly in the middle of the semester, but it is also used to describe situations in which there are several intermediate exams given at different times during the semester.

**Sequence of tests.** Instead of a single exam at the end of the class, let us assume that we have \( m \) exams which are equally spaced in time, at moments
\[ T_1, \ T_2 = 2T_1, \ldots, \ T_m = m \cdot T_1 = T. \]

We consider a simplified model of learning in which there is no forgetting, i.e., in which, once a student learned some subject, the student will retain this knowledge. In this simplified model, once we have checked that a student knows some material, there is no need to check this knowledge again at a later moment of time. It therefore makes sense to arrange the tests in such a way that the \( i \)-th test tests only the knowledge that would normally acquired between the previous test and this one, i.e., between the moments \( T_{i-1} \) and \( T_i \). For the 1-st test, there is no previous test, so we set \( T_0 = 0 \).

**How to determine the grade for each test.** The grade for each test should be determined based on the material that a student should have learned between moments \( T_{i-1} \) and \( T_i \): the ratio of the material that a student has actually learned to the amount that the student could have learned.

**Ideal learning schedule.** In the ideal situation, students study at a uniform speed \( v(t) = v_{\text{ideal}} \). By time \( T \), they should be able to learn all the material, i.e., teach the value
\[ V = \int_0^T v_{\text{ideal}}(t) \, dt = \int_0^T v_{\text{ideal}} \, dt = v_{\text{ideal}} \cdot T = 1. \]

Thus, the ideal learning rate is \( v_{\text{ideal}} = \frac{1}{T} \). With this ideal rate, by the time \( T_{i-1} \), a student would learn the amount
\[ v_{\text{ideal}} \cdot T_{i-1} = \frac{1}{T} \cdot T_{i-1}. \]
In this ideal case, between the times $T_{i-1}$ and $T_i$, the student should have learned the amount $v_{\text{ideal}} \cdot (T_i - T_{i-1})$.

**Actual learning.** The actual amount that a student learns by the time $T_i$ is equal to $\int_0^{T_i} v(t) \, dt$. Thus, if we only count the material that a student should have learned after time $T_{i-1}$, we get the amount

$$\max \left( \frac{T_i}{0} v(t) \, dt - v_{\text{ideal}} \cdot T_{i-1}, 0 \right).$$

(Maximum with 0 is introduced since if a student is behind, i.e., the student have not even started learning the material that he or she was supposed to learn between moments $T_{i-1}$ and $T_i$, the difference will be negative, but the student’s grade cannot be negative, so it 0.)

**Grade for the $i$-th test.** As we have mentioned, the grade $g_i$ for the $i$-th test is equal to the ratio of what the student actually learned to what he or she should have learned, i.e., to the ratio

$$g_i = \frac{\max \left( \frac{T_i}{0} v(t) \, dt - v_{\text{ideal}} \cdot T_{i-1}, 0 \right)}{v_{\text{ideal}} \cdot (T_i - T_{i-1})}.$$

**The overall grade.** The overall grade $g$ is usually computed as a weighted average of grades $g_i$ of different tests, i.e., as a value $g = \sum_{i=1}^m w_i \cdot g_i$, where $w_1, \ldots, w_m$ be the corresponding weights, $w_i \geq 0, \sum_{i=1}^m w_i = 1$.

**Utility caused by the overall grade.** A student learns his or her grade for the $i$-th test practically right after the test, i.e., at the moment $T_i$. Thus, the student’s utility of test $i$ is discounted with a factor $e^{-\alpha \cdot T_i}$.

**Resulting utility.** As a result, the student’s total utility for the learning schedule $v(t)$ is equal to

$$U = \sum_{i=1}^m w_i \cdot e^{-\alpha \cdot T_i}, \quad \max \left( \frac{T_i}{0} v(t) \, dt - v_{\text{ideal}} \cdot T_{i-1}, 0 \right) - \frac{v_{\text{ideal}} \cdot (T_i - T_{i-1})}{v_{\text{ideal}} \cdot (T_i - T_{i-1})}.$$ 

At each moment $t_0$ between the tests $j - 1$ and $j$, the student’s utility is similarly equal to

$$U = \sum_{i=j}^m w_i \cdot e^{-\alpha \cdot (T_i - t_0)}$$

$$\max \left( \frac{T_i}{0} v(t) \, dt - v_{\text{ideal}} \cdot T_{i-1}, 0 \right) - \frac{v_{\text{ideal}} \cdot (T_i - T_{i-1})}{v_{\text{ideal}} \cdot (T_i - T_{i-1})}.$$ 

$$K \cdot \int_0^T e^{-\alpha \cdot (t - T_0)} \cdot v(t) \, dt.$$ 

This is a linear function of the values $v(t)$. A student is encouraged to study at this moment of time $t_0$ if the coefficient at $v(t_0)$ in the above expression is non-negative, i.e., when

$$\frac{w_j}{v_{\text{ideal}} \cdot (T_j - T_{j-1})} \cdot e^{-\alpha \cdot (T_j - T_0)} \geq K.$$ 

We want this inequality to be satisfied for all the values $t_0 \in [T_{j-1}, T_j]$. The right-hand side of the desired inequality is a constant not depending on $t_0$ at all; thus, for this inequality to hold for all $t_0 \in [T_{j-1}, T_j]$, it is sufficient to check that it is true when the left-hand side attains its smallest possible value.

The left-hand side of the desired inequality is an increasing function of $t_0$. Thus, this left-hand side attains its smallest value when the variable $t_i$ attains its smallest possible value $t_0 = T_j$. For this value, the above inequality takes the form

$$\frac{w_j}{v_{\text{ideal}} \cdot (T_j - T_{j-1})} \cdot e^{-\alpha \cdot (T_j - T_0)} \geq K,$$

i.e., equivalently,

$$w_j \geq v_{\text{ideal}} \cdot (T_j - T_{j-1}) \cdot e^{\alpha \cdot T_j - T_{j-1}} \cdot K.$$ 

Since the tests are equally spaced, the right-hand side is the same for all $j$. Let us denote this common value by $w_0$; under this notation, the above inequality takes the form $w_j \geq w_0$ for all $j$, i.e., equivalently, $\min_{j=1,\ldots,m} w_j \geq w_0$.

**Towards optimal selection of the weights corresponding to different tests.** Our objective is to make sure that the testing schedule encourages as many students as possible to study – of course, within the time limitations caused by the fact that a student is taking several classes at the same time.

Different students have different discount parameters $\alpha$. The value $w_0$ depends on the discount parameter $\alpha$; the larger $\alpha$, the larger $w_0$. We can thus say that different students are characterized by different values of the parameter $w_0$. A grading scheme works for a student with the value $w_0$ if and only if $w_0 \leq \min_{j=1,\ldots,m} w_j$. Thus, to make sure that this scheme works for as many students as possible, we must make sure that the threshold $\min_{j=1,\ldots,m} w_j$ is as large as possible.

Since $\sum_{j=1}^m w_j = 1$, we cannot have $\min_{j=1,\ldots,m} w_j > \frac{1}{m}$, since then we would have

$$\sum_{j=1}^m w_j \geq m \cdot \min_{j=1,\ldots,m} w_j > m \cdot \frac{1}{m} = 1.$$
Thus, we should have \( \min w_j \leq \frac{1}{m} \). When all the weights are equal, we have \( w_j = \frac{1}{m} \) and thus, \( \min_{j=1,...,m} w_j = \frac{1}{m} \). Thus, the threshold is the largest when all the weights are equal.

Let us show that equal weights are the only case when the threshold is equal to its largest value \( \frac{1}{m} \). Indeed, if the weights are not equal, one of these weights must be smaller than \( \frac{1}{m} \); otherwise, if all are larger than or equal to \( \frac{1}{m} \) and some are different, the sum will be larger than 1. Thus, we arrive at the following conclusion.

**Conclusion.** The grading scheme is optimal if and only if we assign equal weights to all the midterm exams.

**Comment.** This conclusion is in good accordance with how the class grade is usually calculated in the US universities.

V. DISCUSSION AND FUTURE WORK

**Discussion.** One of the results of our paper is that we proved mathematically what has been known in education for some time – the need for frequent student activities. An instructor has to keep giving quizzes, exams, assignments, etc., to keep the students engaged on the material; otherwise, the students will just wait for the final or the midterm and not do much of anything else to learn the material. That is why it is so important to have learning activities for students.

The need to have learning activities is well understood in both grading systems described in this paper:

- in the US system, where the overall grade for the course is a combination of grades for intermediate assignments, and
- in the Russian system, where the overall grade for the course is the grade for the final exam – provided that a student passed all intermediate assignments with a passing grade.

The difference between these systems is how much weight is assigned to the intermediate assignments (such as midterm exams):

- in the US system, we assign non-zero weights to all the exams, while
- in the Russian system, midterm exams are assigned 0 weight and the final exam the full weight.

Within the US system, different instructors assign different weights to different midterm exams:

- some instructors assign equal weight to all midterms,
- other instructors assign different weights to different midterm exams – e.g., assign more weight to later midterm exams, so that students who did not do well on the first midterm exam still have a chance to learn the material by the second exam and get both a good knowledge and a good grade.

In this paper, we show that the optimal learning occurs when all midterm exams are given non-zero weight – moreover, when different midterm exams are assigned the same weight.

**Future work.** In our approach, we only discusses exams. Many courses use several different kinds of graded student work besides exams, such as quizzes, homeworks, lab assignments, etc. It is desirable to take into account the difference between these types of work.

It is also desirable to not only describe the optimal grading policy for a given schedule of graded work, but also to optimize the schedule itself. For example, it is desirable to find out what is the optimal number of exams for a semester, what is the optimal number of quizzes (and are these quizzes beneficial at all), what emphasize should be placed on the final comprehensive exam – and should we have such an exam at all, etc.

ACKNOWLEDGMENT

The authors are thankful to anonymous referees for valuable suggestions.

REFERENCES


APPENDIX

DISCOUNTING: GENERAL EXPLANATION

Let us show that the formula for discounting can be derived from first principles, without using specifics of financial situations. Indeed, let \( d(t) \) be a discounting after time \( t \), meaning that one unit of utility at a future time \( t \) is equivalent to \( d(t) < 1 \) units at present.

The longer time delay, the less effect the future utility has on the present decisions; thus, the discounting function \( d(t) \) should be decreasing with time.

Let \( s > 0 \) and \( s' > 0 \) be two real numbers. Then, we can describe one unit of utility at time \( s + s' \) in two different ways:

- First, we can directly use the definition of the discounting function \( d(t) \) and conclude that this unit is equivalent to \( d(s + s') \) units at present.
- Alternatively:
  - We can first conclude that 1 unit at time \( s + s' \) is equivalent to \( d(s') \) units \( s \) moments earlier, at the moment \( s \).
  - Then, since one unit at moment \( s \) is equivalent to \( d(s) \) units at present, we conclude that \( d(s') \) units at moment \( s \) are equivalent to \( d(s) \cdot d(s') \) units at present.
Thus, one unit of utility at moment $s + s'$ is equivalent to $d(s) \cdot d(s')$ units at present. From these two conclusions, we deduce that

$$d(s + s') = d(s) \cdot d(s').$$

It is known that every monotonically decreasing solution to this functional equation has the desired form

$$d(t) = \exp(-\alpha \cdot t)$$

for some real value $\alpha > 0$; see, e.g., [1]. Thus, we get the desired justification of the standard discounting formula.