FOR DESCRIBING UNCERTAINTY, ELLIPSOIDS ARE BETTER THAN GENERIC POLYHEDRA AND PROBABLY BETTER THAN BOXES: A REMARK

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For a single quantity, the set of all possible values is usually an interval. An interval is easy to represent in a computer: e.g., we can store its two endpoints. For several quantities, the set of possible values may have an arbitrary shape. An exact description of this shape requires infinitely many parameters, so in a computer, we have to use a finite-parametric approximation family of sets. One of the widely used methods for selecting such a family is to pick a symmetric convex set and to use its images under all linear transformations. If we pick a unit ball, we end up with ellipsoids; if we pick a unit cube, we end up with boxes and parallelepipeds; we can also pick a polyhedron. In this paper, we show that ellipsoids lead to better approximations of actual sets than generic polyhedra; we also show that, under a reasonable conjecture, ellipsoids are better approximators than boxes.

1. Formulation of the Problem

Need for describing sets of possible values. Measurement and estimates are never 100% accurate. As a result, we usually do not know the exact value of a physical quantity; we usually know the set of possible values of this quantity. For a single quantity, this set is usually an interval. Representing an interval in a computer is easy: e.g., we can represent an interval by its endpoints; see, e.g., [7, 10].

For several quantities $x_1, \ldots, x_n$, in addition to interval bounds on each of these quantities, we often have additional restrictions on their combinations; as a result, the set of possible values of $x = (x_1, \ldots, x_n)$ can have different shapes. The space of all possible sets is infinite-dimensional, meaning that we need infinitely many real-valued parameters to represent a generic set. In a computer, at any given time, we can only store finitely many parameters; so, we cannot represent generic sets exactly, we need to approximate them by sets from a finite-parametric family.

Convex set-based representation of sets of possible values. In many practical situations, e.g., when $x_i$ are spatial coordinates, the selection of the quantities is rather arbitrary: we can use a different coordinate system in which, instead
of the original quantities $x_i$, we use linear combinations $y = Tx$, i.e., $y_i = \sum_{j=1}^{m} t_{ij} \cdot x_j$.

In view of this, a reasonable way to select a finite-parametric set is to pick a bounded symmetric convex set $S_0$ with non-empty interior, and to use images $TS_0$ of this set $S_0$ under arbitrary linear transformations $T$.

If we start with a Euclidean unit ball $S_0 = B \overset{def}{=} \left\{ x : \sum_{i=1}^{n} x_i^2 \leq 1 \right\}$, we get the family of ellipsoids (see, e.g., [1–4, 11–14, 16]); if we start with a unit cube $S_0 = C \overset{def}{=} \{ x : |x_i| \leq 1 \text{ for all } i \}$, we get the family of all boxes (plus the corresponding parallelepipeds); alternatively, we can also start with a symmetric convex polyhedron $P$.

**Which set $S_0$ should we choose?** Once we pick a set $S_0$, we can (precisely) represent sets $S$ of the type $TS_0$. If we start with such a set $S$, we enclose it into a set $TS_0 = S$, and then, if we want to enclose $TS_0$ in a set $\lambda \cdot S$ corresponding to the original $S$-based representations, we get the same original set $S = TS_0$ back, with $\lambda = 1$.

For sets $S$ which are different from $TS_0$, the $S_0$-based representation is only approximate. We start with a set $S$, and we enclose it in a set $TS_0 \supseteq S$ for an appropriate linear transformation $T$. If we then try to enclose $TS_0$ in a set of the type $\lambda \cdot S$, then we inevitably get $\lambda > 1$.

The smaller $\lambda$, the better the approximation. It is therefore reasonable, as a measure $d(S_0, S)$ of accuracy of approximating $S$ by $S_0$, to use the smallest $\lambda$ corresponding to all possible $T$:

$$d(S_0, S) = \inf\{ \lambda : \exists T \left( S \subseteq TS_0 \subseteq \lambda \cdot S \right) \}.$$ 

This quantity is known as a *Banach-Mazur distance* between the convex sets $S$ and $S_0$; see, e.g., [15, 17].

For each “standard” set $S_0$, we get different values $A(S_0, S)$ for different sets $S$. As a measure of quality $Q(S_0)$ of choosing $S_0$, it is reasonable to select the worst-case approximation accuracy

$$Q(S_0) \overset{def}{=} \sup_{S} d(S_0, S),$$

where the supremum is taken over all possible bounded symmetric convex sets $S$ with non-empty interior.
2. Main Results

Main conclusion: ellipsoids are better than generic polyhedra. According to the well-known John’s Theorem [8, 15, 17], for the Euclidean unit ball \( B \), we have \( d(B, S) \leq \sqrt{n} \) for all symmetric convex sets \( S \). Thus, we have \( Q(B) \leq \sqrt{n} \).

On the other hand, according to Gluskin’s theorem [6, 15, 17], there exists a constant \( c > 0 \) such that for each dimension \( n \), there exist polyhedra \( P \) and \( P' \) for which \( d(P, P') \geq c \cdot n \) and for which, therefore, \( d(P) \geq c \cdot n \). Moreover, if we take a convex hull \( P \) of 2\( n \) points randomly selected from a unit Euclidean sphere, then, with high probability, we get \( Q(P) \geq c \cdot n \). Since for large \( n \), we have \( c \cdot n \gg \sqrt{n} \) and therefore, \( Q(B) \ll Q(P) \), this shows that for large dimensions, ellipsoids are indeed better than generic polyhedra.

Additional conclusion: ellipsoids are probably better than boxes. A Euclidean unit ball \( B \) (corresponding to ellipsoids) and a unit cube \( C \) (corresponding to boxes) can be viewed as particular cases of unit balls \( B_p \equiv \{ x : \|x\|_p \leq 1 \} \) in the \( \ell_p \)-metric \( \|x\|_p \equiv \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} : B \) is a unit ball in the \( \ell_2 \)-metric while \( C \) is a unit ball in the \( \ell_\infty \)-metric: \( B = B_2 \) and \( C = B_\infty \). The exact values of \( d(B_p, B_q) \) are known only when both \( p \) and \( q \) are on the same side of 2; in this case, \( d(B_p, B_q) = n^{1/p-1/q} \).

In particular, for \( p = 1 \) and \( q = 2 \), we get \( d(B_1, B_2) = \sqrt{n} \).

These values have the property that when \( p < q \), then \( d(B_p, B_q) \) strictly increases when \( p \) decreases or when \( q \) increases; in other words, the larger the difference between \( p \) and \( q \), the larger the value \( d(B_p, B_q) \). For values \( p \) and \( q \) on different sides of 2, this monotonicity does not hold for \( n = 2 \), since in this case, \( B_1 \) (rhombus) and \( B_\infty \) (square) are linearly equivalent and thus, \( d(B_1, B_\infty) = 0 \). However, for \( n > 3 \), we do not have this anomaly and therefore, it is reasonable to conjecture that for \( n > 3 \), this monotonicity holds. Under this hypothesis, \( d(B_\infty, B_1) > d(B_2, B_1) = \sqrt{n} \), and thus, \( Q(B_\infty) \geq d(B_\infty, B_1) > \sqrt{n} \). Since \( Q(B_2) = \sqrt{n} \), we therefore conclude that \( Q(B_2) < Q(B_\infty) \) and thus, ellipsoids are better than boxes.

Comment. These results are in line with a general result according to which, under certain conditions, ellipsoids are the best approximators [5, 9].

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Литература