Picture Fuzzy Sets - a new concept for computational intelligence problems

Bui Cong Cuong*
Institute of Mathematics
Vietnam Academy of Science and Technology
18 Hoang Quoc Viet, Cau Giay, Hanoi, Vietnam
bccuong@gmail.com

Vladik Kreinovich
Department of Computer Science
University of Texas at El Paso
El Paso, TX 79968, USA
vladik@cs.utep.edu

***Abstract*—Since Zadeh introduced fuzzy sets in 1965, a lot of new theories treating imprecision and uncertainty have been introduced. Some of these theories are extensions of fuzzy set theory, other try to handle imprecision and uncertainty in different way. In this paper, we introduce a new notion of picture fuzzy sets (PFS), which are directly extensions of fuzzy sets and of intuitonistic fuzzy sets (Atanassov) . Then some operations on picture fuzzy sets are defined and some properties of these operations are considered. Here the basic preliminaries of PFS theory are presented.**

Keywords: Picture fuzzy set, Operation, Intuitionistic fuzzy set.

I. INTRODUCTION

Computational Intelligence is a set of Nature-inspired computational methodologies and approaches based on the achievements of mathematics, computer science, artificial intelligence, to address complex problems of the real world applications to which traditional methodologies and approaches are ineffective or infeasible.

Naturally, researches about operators and computing processes in fuzzy sets and fuzzy logic promised to contribute to Computational Intelligence, especially to the real word problems, in which there are many vague or uncertainty factors.

Since Zadeh introduced fuzzy sets (FS) in 1965, a lot of new theories treating imprecision and uncertainty have beeen introduced. Some of these theories are extensions of fuzzy set theory, other try to handle imprecision and uncertainty in different way. Intuitionistic fuzzy sets (IFS) were introduced by Atanassov[1] in 1983 constitute a generalization of the notion of a fuzzy set .When fuzzy set give the degree of membership of an element in a given set, intuitionistic fuzzy sets give a degree of membership and a degree of non-membership.

Definition 1:[1] A intuitionistic fuzzy set A on a universe X is an object of the form

$$A=\left\{\left(x,\mu_A(x),\nu_A(x)\right)\middle| x\in X\right\} ,$$

where $\mu_A\left(x\right)(\in\left[0,1\right])$ is called the "degree of membership of x in A", $\nu_A\left(u\right)(\in\left[0,1\right])$ is called the "degree of non-membership of x in A" , and where μ_A and ν_A satisfy the following condition:

$$(\forall x\in X)\quad (\mu_A(x)+\nu_A(x)\le 1) \quad .$$

In this paper, let $IFS(X)$ denote the set of all the intuitionistic fuzzy set IFSs on a universe X **.**

A generalization of fuzzy sets and intuitionistic fuzzy sets are the following notion of picture fuzzy sets .

Definition 2: A picture fuzzy set A on a universe X is an object of the form

$$A=\left\{(x,\mu_A(x),\eta_A(x),\nu_A(x))\middle| x\in X\right\} ,$$

where $\mu_A\left(x\right)(\in\left[0,1\right])$ is called the "degree of positive membership of x in A", $\eta_A(x)(\in[0,1])$ is called the "degree of neutral membership of x in A" and $\nu_A\left(u\right)(\in\left[0,1\right])$ is called the "degree of negative membership of x in A", and where μ_A **,** η_A and ν_A satisfy the following condition:

$$(\forall x\in X)\quad (\mu_A(x)+\eta_A(x)+\nu_A(x)\le 1) \quad .$$

Then *for* $x\in X$, $1-(\mu_A(x)+\eta_A(x)+\nu_A(x))$ could be called the "degree of refusal membership of x in A".

Basically , picture fuzzy sets based models may be adequate in situations when we face human opinions involving more answers of type: yes, abstain, no , refusal.

Voting can be a good example of such a situation as the human voters may be divided into four group of those who: vote for, abstain, vote against, refusal of the voting.

Let $PFS(X)$ denote the set of all the picture fuzzy set PFSs on a universe X **.**

Definition 3: For every two PFSs A and B, the union, intersection and complement are defined as follows:

- $A\subseteq B$ *iff* $(\forall x\in X,\ \mu_A(x)\le\mu_B(x)$ *and* $\eta_A(x)\le\eta_B(x)$ *and* $\nu_A(x)\ge\nu_B(x))$
- $A=B$ *iff* $(A\subseteq B$ *and* $B\subseteq A)$

* Corresponding author. Email: bccuong@gmail.com.
Phone: 84-904211348
Digital Object Identifier inserted by IEEE

- $A \cup B = \{(x, \max(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \min(\nu_A(x), \nu_B(x))) | x \in X\}$
- $A \cap B = \{(x, \min(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \max(\nu_A(x), \nu_B(x))) | x \in X\}$
- $coA = \overline{A} = \{(x, \nu_A(x), \eta_A(x), \mu_A(x)) | x \in X\}$.

Now we will introduce a generalisation of interval-valued fuzzy set A . Here the $\text{int}([0,1])$ stands for the set of all closed subinterval of $[0,1]$.

Definition 4: Let $[a_1, b_1], [a_2, b_2] \in \text{int}([0,1])$. We define

$$[a_1, b_1] \le [a_2, b_2], \quad iff\ a_1 \le a_2, b_1 \le b_2$$

$$[a_1, b_1] \precsim [a_2, b_2] \quad iff\ a_1 \le a_2, b_1 \ge b_2$$

$$[a_1, b_1] = [a_2, b_2], \quad iff\ a_1 = a_2, b_1 = b_2$$

$$[a_1, b_1] \vee [a_2, b_2] = [a_1 \vee a_2, b_1 \vee b_2]$$

$$[a_1, b_1] \wedge [a_2, b_2] = [a_1 \wedge a_2, b_1 \wedge b_2]$$

$$[a_1, b_1] \times [a_2, b_2] = [a_1 \times a_2, b_1 \times b_2]$$

Let $\alpha \ge 0$. We define $\alpha[a_1, b_1] = [\alpha a_1, \alpha b_1]$.

If $0 \le \alpha \le 1$, then $\alpha[a_1, b_1] \in \text{int}([0,1])$.

Definition 5: An interval-valued picture fuzzy set A on a universe X (IvPFS, in short) is an object of the form

$$A = \{(x, M_A(x), L_A(x), N_A(x)) | x \in X\} ,$$

where $M_A(x) = [M_{AL}(x), M_{AU}(x)] \in \text{int}([0,1])$,

$L_A(x) = [L_{AL}(x), L_{AU}(x)] \in \text{int}([0,1])$,

$N_A(x) = [N_{AL}(x), N_{AU}(x)] \in \text{int}([0,1])$,

satisfy the following condition:

$$(\forall x \in X) \quad (\sup M_A(x) + \sup L_A(x) + \sup N_A(x) \le 1) .$$

Let $IvPFS(X)$ denote the set of all the interval-valued picture fuzzy set IvPFSs on a universe X .

Definition 6: For every two IvPFSs A and B, the inclusion, union, intersection and complement are defined as follows:

- $A \subseteq B\ iff\ (\forall x \in X)((M_A(x) \le M_B(x))\ and\ (L_A(x) \le L_B(x))\ and\ (N_A(x) \ge N_B(x)))$
- $A = B\ iff\ A \subseteq B\ and\ B \subseteq A$
- $A \cup B = \{(x, M_A(x) \vee M_B(x), L_A(x) \wedge L_B(x), N_A(x) \wedge N_B(x)) | x \in X\}$
- $A \cap B = \{(x, M_A(x) \wedge M_B(x), L_A(x) \wedge L_B(x), N_A(x) \vee N_B(x)) | x \in X\}$

where $\vee$ and $\wedge$ stand for max and min operators respectively

- $coA = \overline{A} = \{(x, N_A(x), L_A(x), M_A(x)) | x \in X\}$.

Definition 7: Let X_1 and X_2 be two universums and let $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) | x \in X_1\}$ and $B = \{(y, \mu_B(y), \eta_B(y), \nu_B(y)) | y \in X_2\}$ be two PFSs .

We define the Cartesian product of these two PFS's

- $A \times_1 B = \left\{ \begin{array}{l} ((x,y), \mu_A(x).\mu_B(y), \eta_A(x).\eta_B(y), \\ \nu_A(x).\nu_B(y)) | x \in X_1, y \in X_2 \end{array} \right\}$
- $A \times_2 B = \left\{ \begin{array}{l} ((x,y), \mu_A(x) \wedge \mu_B(y), \eta_A(x) \wedge \eta_B(y), \\ \nu_A(x) \vee \nu_B(y)) | x \in X_1, y \in X_2 \end{array} \right\}$

These definitions are valid.

Proof: See [7] .

Definition 8: Let A be an IvPFS over X_1 and B be an IvPFS over X_2 .We define:

- $A \times_1 B = \{((x,y), (M_A(x) \times M_B(y), L_A(x) \times L_B(y), N_A(x) \times N_B(y)) | x \in X_1, y \in X_2\}$
- $A \times_2 B = \{((x,y), (M_A(x) \wedge M_B(y), L_A(x) \wedge L_B(y), N_A(x) \vee N_B(y)) | x \in X_1, y \in X_2\}$

The proof of the validity of these operators is given in [7].

Some of the defined operations are extensions of the FS operations [6], [11] , [12] and of some of operations for IFSs and IvIFSs [1 – 4], [10] and [13].

II. SOME OPERATIONS ON PFS

Now we consider some propeties of the defined operations on PFS.

Proposition 1: For every PFS's A,B,C

(a) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$;

(b) $\overline{(\overline{A})} = A$;

(c) Operations $\cap$ and $\cup$ are commutative , associative, and distributive;

(d) Operations $\cap$, Co and $\cup$ satisfy the law of De Morgan.

Proof: See [7].

Proposition 2: For every three universums X_1, X_2, X_3 and four PFS's (or four IvPFS's) A, B (over X_1), C (over X_2), D (over X_3):

(a) $A \times_1 C = C \times_1 A$;

(b) $((A \times_1 C) \times_1 D = A \times_1 (C \times_1 D)$;

(c) $(A \cup B) \times_1 C = (A \times_1 C) \cup (B \times_1 C)$;

(d) $(A \cap B) \times_1 C = (A \times_1 C) \cap (B \times_1 C)$.

Proof: See [7].

Proposition 3: For the Cartesian produce of PFSs

$$A \times_2 B = \left\{ \begin{matrix} ((x,y), \mu_A(x) \wedge \mu_B(y), \eta_A(x) \wedge \eta_B(y), \\ \nu_A(x) \vee \nu_B(y)) \,|\, x \in X_1, y \in X_2 \end{matrix} \right\}$$

analogously, we have

(a) $A \times_2 C = C \times_2 A$;

(b) $((A \times_2 C) \times_2 D = A \times_2 (C \times_2 D)$;

(c) $(A \cup B) \times_2 C = (A \times_2 C) \cup (B \times_2 C)$;

(d) $(A \cap B) \times_2 C = (A \times_2 C) \cap (B \times_2 C)$

Proof: See [8].

Proposition 4: For every three universums X_1, X_2, X_3 and four IvPFS's A, B (over X_1), C (over X_2), D (over X_3). For the Cartesian produce of IvPFSs

$$A \times_2 B = \left\{ \begin{matrix} ((x,y), M_A(x) \wedge M_B(y), L_A(x) \wedge L_B(y), \\ N_A(x) \vee N_B(y)) \,|\, x \in X_1, y \in X_2 \end{matrix} \right\}$$

analogously, we have:

(a) $A \times_2 C = C \times_2 A$;

(b) $((A \times_2 C) \times_2 D = A \times_2 (C \times_2 D)$;

(c) $(A \cup B) \times_2 C = (A \times_2 C) \cup (B \times_2 C)$;

(d) $(A \cap B) \times_2 C = (A \times_2 C) \cap (B \times_2 C)$

Proof: See [8].

A. Distance between picture fuzzy sets

Distance between fuzzy sets and distance between intuitionistic fuzzy sets were defined in fuzzy literature and have been applied in various problems [10],[11]. In this sub-section some extensions of the distances between intuitionistic fuzzy sets of Szmith and Kacprzyk [13] are presented.

Definition 9: Distances for two picture fuzzy sets A and B in $X = \{x_1, x_2, ..., x_n\}$ are:

- The normalized Hamming distance $d_P(A,B)$

$$d_P(A,B) = \frac{1}{n}\sum_{i=1}^{n}(|\mu_A(x_i) - \mu_B(x_i)| + |\eta_A(x_i) - \eta_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|)$$

- The normalized Euclidean distance $e_P(A,B)$

$$e_P(A,B) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}((\mu_A(x_i)) - \mu_B(x_i))^2 + (\eta_A(x_i) - \eta_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2)}$$

Example 1: Let us consider picture fuzzy sets A, B, C in $X = \{x_1, x_2, x_3\}$, A full description of picture fuzzy set A, i.e. $A = \{(\mu_A(x_1), \eta_A(x_1), \nu_A(x_1))/x_1, (\mu_A(x_2), \eta_A(x_2), \nu_A(x_2))/x_2, (\mu_A(x_3), \eta_A(x_3), \nu_A(x_3))/x_3\}$.

For example,

$A = \{(0.8, 0.1, 0)/x_1, (0.4, 0.2, 0.3)/x_2, (0.5, 0.3, 0)/x_3\}$

$B = \{(0.3, 0.3, 0.2)/x_1, (0.7, 0.1, 0.1)/x_2, (0.4, 0.3, 0.2)/x_3\}$

$C = \{(0.3, 0.4, 0.1)/x_1, (0.6, 0.2, 0.1)/x_2, (0.4, 0.3, 0.1)/x_3\}$.

Then $d_P(A,B) = 0.6$, $d_P(B,C) = 0.5/3$, $d_P(A,C) = 0.5$

$e_P(A,B) = \sqrt{0.52/3}$, $e_P(B,C) = \sqrt{0.05/3}$, $e_P(A,C) = \sqrt{0.15}$.

B. Convex combination of PFS

Convex combination is an important operation in mathematics, which is an useful tool on convex analysis, linear spaces and convex optimization. In this sub-section convex combination firstly is defined with some simple propositions. This new operation should be more considered in the other papers.

Definition 10: Let A , B be a PFS on X . Let θ be a real number such that $0 \le \theta \le 1$.

For each θ, the convex combination of A and B is defined as follows:

$C_\theta(A,B) = \left\{\left(x, \mu_{C_\theta}(x), \eta_{C_\theta}(x), \nu_{C_\theta}(x)\right) \,|\, x \in X\right\}$ where

$\forall x \in X$, $\mu_{C_\theta}(x) = \theta.\mu_A(x) + (1-\theta).\mu_B(x)$,

$\forall x \in X$, $\eta_{C_\theta}(x) = \theta.\eta_A(x) + (1-\theta).\eta_B(x)$,

$\forall x \in X$, $\nu_{C_\theta}(x) = \theta.\nu_A(x) + (1-\theta).\nu_B(x)$.

Analogously, the convex combination of IvPFSs is defined.

Definition 11: Let A, B be two IvPFS on X. Let θ be a real number such that $0 \le \theta \le 1$.

For each θ, the convex combination of A and B is defined as follows:

$$C_\theta(A,B) = \left\{\left(x, M_{C_\theta}(x), L_{C_\theta}(x), N_{C_\theta}(x)\right) \middle| x \in X\right\}$$

where

$\forall x \in X$, $M_{C_\theta}(x) = [(M_{C_\theta L}(x) = \theta.M_{AL}(x) + (1-\theta).M_{BL}(x)),$
$(M_{C_\theta U}(x) = \theta.M_{AU}(x) + (1-\theta).M_{BU}(x))],$

$\forall x \in X$, $L_{C_\theta}(x) = [(L_{C_\theta L}(x) = \theta.L_{AL}(x) + (1-\theta).L_{BL}(x)),$
$(L_{C_\theta U}(x) = \theta.L_{AU}(x) + (1-\theta).L_{BU}(x))],$

$\forall x \in X$, $N_{C_\theta}(x) = [(N_{C_\theta L}(x) = \theta.N_{AL}(x) + (1-\theta).N_{BL}(x)),$
$(N_{C_\theta U}(x) = \theta.N_{AU}(x) + (1-\theta).N_{BU}(x))].$

Proposition 5: Let A, B be two PFS (or two IvPFS) on X. Let θ be a real number such that $0 \le \theta \le 1$.Then

- If $\theta = 1$, then $C_\theta(A,B) = A$ and if $\theta = 0$, then $C_\theta(A,B) = B$;
- If $A \subseteq B$, then $\forall \theta$, $A \subseteq C_\theta(A,B) \subseteq B$;
- If $(A \supseteq B) \& (\theta_1 \ge \theta_2)$, then $C_{\theta_1}(A,B) \supseteq C_{\theta_2}(A,B)$.

Proof: See [8].

Proposition 6: Let A, B, D be three PFS (or three IvPFS) on X. Let θ be a real number such that $0 \le \theta \le 1$.Then

- $C_\theta(A \cap B, D) = C_\theta(A,D) \cap C_\theta(B,D)$;
- $C_\theta(A \cup B, D) = C_\theta(A,D) \cup C_\theta(B,D)$.

Proof: See [8].

We can define some simple aggregation operators on PFS.

Definition 12: Let $\{A_1,...,A_n\}$ be a vector of PFSs on X, where $A_i = \left\{\left(x, \mu_{A_i}(x), \eta_{A_i}(x), \nu_{A_i}(x)\right) \middle| x \in X\right\}$, i=1,,,,,n

Let $\alpha = \{\alpha_1,...,\alpha_n\}$ be a weighted vector such that $0 \le \alpha_i \le 1$, i=1,...,n and $\sum_i \alpha_i = 1$.

The weighted mean of vector $\{A_1,...,A_n\}$ is the following

$$C(\alpha, A) = \left\{\left(x, \mu_{C(\alpha,A)}(x), \eta_{C(\alpha,A)}(x), \nu_{C(\alpha,A)}(x)\right) \middle| x \in X\right\}$$

where $\forall x \in X$, $\mu_{C(\alpha,A)}(x) = \sum_{i=1}^{n} \alpha_i.\mu_{A_i}(x)$,

$\forall x \in X$, $\eta_{C(\alpha,A)}(x) = \sum_{i=1}^{n} \alpha_i.\eta_{A_i}(x)$,

$\forall x \in X$, $\nu_{C(\alpha,A)}(x) = \sum_{i=1}^{n} \alpha_i.\nu_{A_i}(x)$.

When $\alpha = \alpha_0 = \{1/n,...,1/n\}$, we obtain the arithmetic mean of vector $\{A_1,...,A_n\}$

$$C(\alpha_0, A) = \left\{\left(x, \mu_{C(\alpha_0,A)}(x), \eta_{C(\alpha_0,A)}(x), \nu_{C(\alpha_0,A)}(x)\right) \middle| x \in X\right\}$$

where $\forall x \in X$, $\mu_{C(\alpha_0,A)}(x) = \frac{1}{n}(\sum_{i=1}^{n} \mu_{A_i}(x))$,

$\eta_{C(\alpha_0,A)}(x) = \frac{1}{n}(\sum_{i=1}^{n} \eta_{A_i}(x))$, $\nu_{C(\alpha_0,A)}(x) = \frac{1}{n}(\sum_{i=1}^{n} \nu_{A_i}(x))$.

Analogously, we can define the weighted mean and the arithmetic mean of vector of IvPFS.

These aggregation operators would be used in the multi-criteria decision making problems.

III. PICTURE FUZZY RELATIONS

Fuzzy relations are one of most important notions of fuzzy set theory and fuzzy systems theory. The Zadeh' composition rule of inference [6],[11] is a well-known method in approximation theory and inference methods in fuzzy control theory. Intuitionistic fuzzy relations were received many results by researches [4], [5] and [10]. Xu [14] defined some new intuitionistic preference relations, such as the consistent intuitionistic preference relation, imcomplete intuitionistic preference relation and studied their properties. Thus, it is necessary to develop new approaches to issues, such as multi-period investment decision making, medical diagnosis, personnel dynamic examination, and military system efficiency dynamic evaluation. In this section we shall present some preliminary results on picture fuzzy relations.

A. Some definitions for intuitionistic relation

Let X , Y and Z be ordinary non-empty sets.

Definition 13: [4] An intuitionistic fuzzy relation is an intuitionistic fuzzy subset of $X \times Y$, i.e. is an expression given by

$$R = \left\{((x,y), \mu_R(x,y), \nu_R(x,y)) \middle| x \in X, y \in Y\right\} ,$$

where $\mu_R : X \times Y \to [0,1]$, $\nu_R : X \times Y \to [0,1]$ satisfy the following condition:
$0 \le \mu_R(x,y) + \nu_R(x,y) \le 1$ for every $(x,y) \in (X \times Y)$.

The most important properties of intuitionistic fuzzy relations were studied in [4],[5].

An extension for picture fuzzy relations is defined as follows:

Definition 14: A picture fuzzy relation R is a picture fuzzy subset of $X \times Y$, i.e. R given by

$$R = \left\{((x,y), \mu_R(x,y), \eta_R(x,y), \nu_R(x,y)) \middle| x \in X, y \in Y\right\}$$

where $\mu_R : X \times Y \to [0,1]$, $\eta_R : X \times Y \to [0,1]$, $\nu_R : X \times Y \to [0,1]$ satisfy the condition $\mu_R(x,y) + \eta_R(x,y) + \nu_R(x,y) \le 1$, for every $(x,y) \in (X \times Y)$.

We will represent by $PFR(X \times Y)$ the set of all the picture fuzzy subsets on $X \times Y$.

Defintion 15: Let $R \in PFR(X \times Y)$. We define the inverse relation R^{-1} between Y and X :
$\mu_{R^{-1}}(y,x) = \mu_R(x,y)$, $\eta_{R^{-1}}(y,x) = \eta_R(x,y)$, $\nu_{R^{-1}}(y,x) = \nu_R(x,y)$, $\forall (x,y) \in (X \times Y)$.

Now we will consider simple properties of PFRs.

Definition 16: Let R and P be two picture fuzzy relations between X and Y , for every $(x,y) \in X \times Y$ we can define:

- $R \le P \Leftrightarrow (\mu_R(x,y) \le \mu_P(x,y))$ & $(\eta_R(x,y) \le \eta_P(x,y))$ & $(\nu_R(x,y) \ge \nu_P(x,y))$
- $R \vee P = \{((x,y), \mu_R(x,y) \vee \mu_P(x,y), \eta_R(x,y) \wedge \eta_P(x,y), \nu_R(x,y) \wedge \nu_P(x,y)) | x \in X, y \in Y\}$
- $R \wedge P = \{((x,y), \mu_R(x,y) \wedge \mu_P(x,y), \eta_R(x,y) \wedge \eta_P(x,y), \nu_R(x,y) \vee \nu_P(x,y)) | x \in X, y \in Y\}$
- $R_c = \{((x,y), \nu_R(x,y), \eta_R(x,y), \mu_R(x,y)) | x \in X, y \in Y\}$

Proposition 7: Let R , P , Q $\in PFR(X \times Y)$. Then

(a) $(R^{-1})^{-1} = R$, (b) $R \le P \Rightarrow R^{-1} \le P^{-1}$

(c1) $(R \vee P)^{-1} = R^{-1} \vee P^{-1}$, (c2) $(R \wedge P)^{-1} = R^{-1} \wedge P^{-1}$

(d1) $R \wedge (P \vee Q) = (R \wedge P) \vee (R \wedge Q)$,

(d2) $R \vee (P \wedge Q) = (R \vee P) \wedge (R \vee Q)$

(e) $R \wedge P \le R$, $R \wedge P \le P$

(f 1) If $(R \ge P)$ & $(R \ge Q)$ then $R \ge P \vee Q$,

(f 2) If $(R \le P)$ & $(R \le Q)$ then $R \le P \wedge Q$.

Proof: See [8].

B. Composition of Picture Fuzzy Relations

In this sub-section we shall present some compositons of PFRs. We shall discuss this important problem for approximation reasoning and fuzzy control in the other papers.
The composition of intuitionistic fuzzy relations is given as follows:

Definition 17: [4] Let $\alpha, \beta, \lambda, \rho$ be t-norms or t-conorms not necessarily dual two-two, $E \in IFR(X \times Y)$ and $P \in IFR(Y \times Z)$. We will call composed relation $P \mathbb{C} E \in IFR(X \times Z)$ to the one defined by

$$P \mathbb{C} E = \left\{((x,z), \mu_{P\mathbb{C}E}(x,y), \nu_{P\mathbb{C}E}(x,z)) \middle| x \in X, z \in Z\right\} ,$$

where $\mu_{P\mathbb{C}E}(x,z) = \underset{y}{\alpha}\{\beta[\mu_E(x,y), \mu_P(y,z)]\}$,

$\nu_{P\mathbb{C}E}(x,z) = \underset{y}{\lambda}\{\rho[\nu_E(x,y), \nu_P(y,z)]\}$

whenever
$0 \le \mu_{P\mathbb{C}E}(x,z) + \nu_{P\mathbb{C}E}(x,z) \le 1 \quad \forall (x,z) \in X \times Z$.

In [4] it was proved that take $\alpha = \vee$, β t-norm , $\lambda = \wedge$, ρ t-conorm , the composition of intuitionistic fuzzy relations satisfies the largest number of properties.

A composition of picture fuzzy relations is defined as follows:

Definition 18: Let $\alpha = \vee$, β t-norm , $\lambda = \wedge$, ρ t-conorm be t-norms or t-conorms not necessarily dual two-two, $E \in PFR(X \times Y)$ and $P \in PFR(Y \times Z)$. We will call composed relation $P \mathbb{C} E \in PFR(X \times Z)$ to the one defined by

$$P \mathbb{C} E = \left\{ \begin{array}{l} ((x,z), \mu_{P\mathbb{C}E}(x,y), \eta_{P\mathbb{C}E}(x,z), \\ \nu_{P\mathbb{C}E}(x,z)) | x \in X, z \in Z) \end{array} \right\} , \text{ where}$$

$\mu_{P\mathbb{C}E}(x,z) = \underset{y}{\vee}\{\beta[\mu_E(x,y), \mu_P(y,z)]\}$,

$\eta_{P\mathbb{C}E}(x,z) = \underset{y}{\vee}\{\beta[\eta_E(x,y), \eta_P(y,z)]\}$,

$\nu_{P\mathbb{C}E}(x,z) = \underset{y}{\wedge}\{\rho[\nu_E(x,y), \nu_P(y,z)]\}$,

whenever
$0 \le \mu_{P\mathbb{C}E}(x,z) + \eta_{P\mathbb{C}E}(x,z) + \nu_{P\mathbb{C}E}(x,z) \le 1, \forall (x,z) \in X \times Z$.

The first composition of PFRs is the generalized max-min compostion in fuzzy set theory.

Definition 19: Let $E \in PFR(X \times Y)$ and $P \in PFR(Y \times Z)$. We will call max-min composed relation $P \mathbb{C} E \in PFR(X \times Z)$ to the one defined by

$$P \mathbb{C} E = \left\{ ((x,z), \mu_{P\mathbb{C}E}(x,y), \eta_{P\mathbb{C}E}(x,z), \nu_{P\mathbb{C}E}(x,z)) \middle| x \in X, z \in Z \right\}$$

where $\mu_{P\mathbb{C}E}(x,z) = \underset{y}{\vee}\{[\mu_E(x,y) \wedge \mu_P(y,z)]\}$,

$$\eta_{P\mathbb{C}E}(x,z) = \underset{y}{\vee}\{[\eta_E(x,y) \wedge \eta_P(y,z)]\} ,$$

$$\nu_{P\mathbb{C}E}(x,z) = \underset{y}{\wedge}\{[\nu_E(x,y) \vee \nu_P(y,z)]\} .$$

The second composition of PFRs is the generalized max-prod composition in fuzzy set theory.

Definition 20: Let $E \in PFR(X \times Y)$ and $P \in PFR(Y \times Z)$. We will call max-prod composed relation $P \mathbb{C} E \in PFR(X \times Z)$ to the one defined by

$$P \mathbb{C} E = \left\{ ((x,z), \mu_{P\mathbb{C}E}(x,y), \eta_{P\mathbb{C}E}(x,z), \nu_{P\mathbb{C}E}(x,z)) \middle| x \in X, z \in Z \right\}$$

, where $\forall (x,z) \in X \times Z$

$$\mu_{P\mathbb{C}E}(x,z) = \underset{y}{\vee}\{[\mu_E(x,y).\mu_P(y,z)]\} ,$$

$$\eta_{P\mathbb{C}E}(x,z) = \underset{y}{\vee}\{[\eta_E(x,y).\eta_P(y,z)]\} ,$$

$$\nu_{P\mathbb{C}E}(x,z) = \underset{y}{\wedge}\{[\nu_E(x,y) + \nu_P(y,z) - \nu_E(x,y).\nu_P(y,z)]\} .$$

IV.CONCLUSIONS

In this paper, the new notion of picture fuzzy sets was introduced. Then some operations on PFSs and some properties of these operations were presented. In the parts 2 and 3 more operations including the convex combination were proposed and picture fuzzy relations with their compositions were firstly discussed. In the following paper some classes of aggregation operations with applications in decision making problems and picture fuzzy logic operators should be considered.

ACKNOWLEDGMENT

This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 102.01-2012.14

REFERENCES

[1] K. Atanassov, "Intuitionistic fuzzy sets", Fuzzy Sets and Systems, vol.20 ,1986, pp. 87-96.

[2] K. Atanassov, "More on intuitionistic fuzzy sets", Fuzzy Sets and Systems, vol.33 ,1989, pp. 37-45.

[3] K. Atanassov and G. Gargov, "Interval valued intuitionistic fuzzy sets", Fuzzy Sets and Systems, vol.31 ,1989, pp. 343-349.

[4] P. Burilo and H. Bustince , "Intuitionistic fuzzy relations. (Part I)", Mathware and Soft Computing , vol.2, 1995, pp. 5-38.

[5] H. Bustince , "Construction of intuitionistic fuzzy relations with predetermined properties", Fuzzy Sets and Systems, vol.109, 1996, pp.379-403.

[6] B.C. Cuong and N.D. Phuoc (Ed.), Fuzzy Systems, Neural Networks and Applications, .Science and Technology, Second Ed., 2006 (in Vietnamese).

[7] B.C.Cuong, "Picture fuzzy sets – first results. Part 1", Seminar " Neuro-Fuzzy Systems with Applications", Preprint 03/2013, Institute of Mathematics, Hanoi, May 2013.

[8] B.C.Cuong, "Picture fuzzy sets – first results. Part 2", Seminar " Neuro-Fuzzy Systems with Applications", Preprint 04/2013, Institute of Mathematics, Hanoi, June 2013.

[9] B.C. Cuong, T.H.Anh and B.D.Hai , "Some operations on type-2 intuitionistic fuzzy sets", J. Computer Science and Cybernetics, vol.28, 2012, pp. 274-283.

[10] G. Deschrijver and E.Kerre , "On the composition of intuitionistic fuzzy relations", Fuzzy Sets and Systems, vol.136,2003, pp.333-361.

[11] J. Fodor and M.Roubens, Fuzzy preference modeling and multicriteria decision support, Kluwer Academic Pub., London,1994.

[12] T.H.Nguyen and E.Walker, A first course in fuzzy logic, Chapman& Hall/CRC, Boca Raton, 2000.

[13] E. Szmidt and J. Kacprzyk, "Distances between intuitionistic fuzzy sets", Fuzzy Sets and Systems. vol.114, 2000, pp.505-518.

[14] Z.S. Xu, "Intuitionistic preference relations and their application in group decision making", Information Science, vol.177,2007,pp.2363-2379

[15] L.A.Zadeh, "Fuzzy Sets", Information and Control , vol.8, 1965, pp. 338-353.