

FUZZY INTERVALS AS FOUNDATION OF METROLOGICAL SUPPORT FOR COMPUTATIONS WITH INACCURATE DATA

SEMENOV, K. K., SOLOPCHENKO, G. N.

Department of Measurement Informational Technologies, St. Petersburg State Polytechnical University, 29, Polytechnicheskaya str., St. Petersburg, 195251, Russia

KREINOVICH, V. YA.

Department of Computer Science, University of Texas at El Paso, 500 W. University, El Paso, TX 79968, USA

In this paper, we discuss the possibility of using the formalism of fuzzy intervals as a basis for computational metrology. We consider advantages of using fuzzy intervals instead of the traditional intervals as a characteristic of uncertainty of the results of computations with inaccurate data.

Measurement is the only way to obtain objective quantitative information about parameters and characteristics of surrounding world. Result of any measurement is inaccurate; that's why when we numerically process obtained data, we get only uncertain results. Characteristics of this uncertainty should be expressed in quantitative form and kept with result itself [1, 2].

There are many different approaches for representing this uncertainty. These approaches take into account different information about the initial data inaccuracy. Some of these approaches use random variables [3-5], other approaches use bounds on possible values of true initial data [6-8]. All these approaches take into account objective information about the initial data uncertainty. Within each of these approaches, there are methods which enable us to use this information to automatically estimate the uncertainty characteristics of results of computations.

A recent tendency of using expert information to solve different metrological problems leads to situations when *a priori* information can be subjective. Approaches which were designed to handle objective data are not well suited for processing such subjective information. Because of this, we propose to use fuzzy set theory – which can process both objective and subjective (expert) data [9].

Let $\tilde{x}_1 = x_1 + \Delta x_1, \dots, \tilde{x}_n = x_n + \Delta x_n$ be the measurement results for quantities x_1, \dots, x_n that were obtained with absolute errors $\Delta x_1, \dots, \Delta x_n$. Let $y = f(x_1, \dots, x_n)$ be the function that describes the necessary computations. As

its result, not only the value $\tilde{y} = f(\tilde{x}_1, \dots, \tilde{x}_n) = f(x_1 + \Delta x_1, \dots, x_n + \Delta x_n)$ should be computed, but also characteristics of its inaccuracy:

$$\Delta y = f(x_1 + \Delta x_1, x_2 + \Delta x_2, \dots, x_n + \Delta x_n) - f(x_1, x_2, \dots, x_n).$$

If the errors $\Delta x_1, \dots, \Delta x_n$ are small, then we can simplify the problem by linearizing of the function $\tilde{y} = f(\tilde{x}_1, \dots, \tilde{x}_n)$. In this case, the resulting inaccuracy becomes a linear combination of the errors $\Delta x_1, \dots, \Delta x_n$:

$$\Delta y \approx \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n) \cdot \Delta x_i. \quad (1)$$

Since the computation of f is performed by a computer program, we can estimate derivatives in Eq. 1 efficiently and accurately with techniques of automatic differentiation [10]. This technique is used in [11, 12] for solving a wide class of metrological problems.

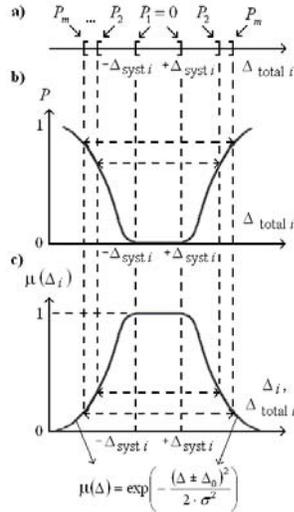


Fig. 1. Membership function construction for fuzzy interval

holds with probability $P > P_{\text{syst}} \cdot P_{\text{rand}} = P_{\text{rand}}$. Hence, the value $\Delta_{\text{total } i}$ of the total error bound is the function of confidence probability P : $\Delta_{\text{total } i} = \Delta_{\text{total } i}(P)$. If we associate the set of intervals $J_{1-P} = [-\Delta_{\text{total } i}(P), \Delta_{\text{total } i}(P)]$ with values $\alpha = 1 - P$ then the received curve $\alpha = \alpha(\Delta_{\text{total } i})$ will correspond to membership function $\alpha = \mu(\Delta_i)$ of a fuzzy interval that will represent information about total error (Fig. 1).

The curve $\mu(\Delta_i)$ is the symmetrical curvilinear trapezoid. Its upper base

Let us describe errors $\Delta x_1, \dots, \Delta x_n$ as fuzzy variables. Operations of addition and multiplication with constant in Eq. 1 should be treated accordingly. Errors $\Delta x_1, \dots, \Delta x_n$ are composed of systematic $\Delta_{\text{syst}} x_1, \dots, \Delta_{\text{syst}} x_n$ and random $\Delta_{\text{rand}} x_1, \dots, \Delta_{\text{rand}} x_n$ components. It should be taken into account that they act differently when we perform multiple measurements.

Usually, it is known from the technical documentation for measuring instruments that $|\Delta_{\text{syst}} x_i| \leq \Delta_{\text{syst } i}$ with probability $P_{\text{syst}} = 1$ and that $|\Delta_{\text{rand}} x_i| \leq \Delta_{\text{rand } i}$ with probability greater than or equal to $P_{\text{rand}} < 1$. So, inequality $|\Delta x_i| = |\Delta_{\text{syst}} x_i + \Delta_{\text{rand}} x_i| \leq \Delta_{\text{total } i} = \Delta_{\text{syst } i} + \Delta_{\text{rand } i}$

represents information about the systematic part of error and its lateral sides describe known information about the error's random component. The value α is the degree of belief of the statement "total error Δx_i of measurement result \tilde{x}_i will be inside the interval J_α ".

In [13, 14], it is theoretically justified that the trapezoid $\alpha = \mu(\Delta_i)$ should have left and right halves of Gaussian curve as its lateral sides (Fig. 1c). If experts produce membership function of another type then it can be easily approximated with function $\tilde{\mu}(\Delta_i)$ of the necessary form. In the report, such algorithm and examples of corresponding approximations will be presented. So, membership function of fuzzy interval can be described with only two parameters $\{\Delta_0, \sigma\}$.

To process fuzzy intervals, different arithmetic operations can be used. In this report, we use results from triangular norms theory to justify a certain class of algebraic bounded operations. If we use algebraic-type arithmetic operations then, to process fuzzy intervals, we process only tuples $\{\Delta_0, \sigma\}$ [15]. Linear operations with fuzzy intervals, which are used in Eq. 1, will lead to:

$$\{\Delta_{01}, \sigma_1\} \pm \{\Delta_{02}, \sigma_2\} = \{\Delta_{01} + \Delta_{02}, \sqrt{\sigma_1^2 + \sigma_2^2}\}, \quad c \cdot \{\Delta_{01}, \sigma_1\} = \{|c| \cdot \Delta_{01}, |c| \cdot \sigma_1\}.$$

We can see that these rules repeat well-known rules that are used in metrology for processing systematic errors and for standard deviations of random errors. From [16, 17], we can conclude that we should use values $\alpha = 0.05 \div 0.10$ to get confidence interval from fuzzy interval. Also, as it was demonstrated in [15], averaging of fuzzy intervals for multiple measurements results reduces the uncertainty of its borders and makes it tend to the classical deterministic interval -- in full correspondence with traditional metrology.

Fuzzy interval description of measurement inaccuracy is in good agreement with known approaches [3-8]. In the report, it is shown that the fuzzy intervals formalism is in good accordance with probabilistic [3] and interval [6] arithmetics. The examples and rules will be presented for constructing fuzzy interval from the empirical data and also from other frameworks for uncertainty.

From the results of paper [14], it can be concluded that the natural domain for using fuzzy intervals is limited to linear operations. Attempts of their using for nonlinear transforms lead to difficulties. That's why it is reasonable to use fuzzy intervals jointly with automatic differentiation techniques.

Let us turn back to Eq. 1. The partial derivatives $\frac{\partial f}{\partial x_i}$ of the function f are computed for values $\tilde{x}_1, \dots, \tilde{x}_n$ -- which are inaccurate. Thus, the value of Δy can be underestimated because we use linearization for f in a slightly different domain: instead of $[x_1 - \Delta_1, x_1 + \Delta_1] \times \dots \times [x_n - \Delta_n, x_n + \Delta_n]$ we take the domain $[\tilde{x}_1 - \Delta_1, \tilde{x}_1 + \Delta_1] \times \dots \times [\tilde{x}_n - \Delta_n, \tilde{x}_n + \Delta_n]$. To take this fact into account, we can use the following approach. If the automatic differentiation is

used for estimating first-order derivatives, then we can apply this technique again (recursively) to obtain the values of the second-order derivatives. If the errors $\Delta x_1, \dots, \Delta x_n$ are small, then the following inequality holds:

$$\left| \frac{\partial f(x_1, x_2, \dots, x_n)}{\partial x_i} - \frac{\partial f(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)}{\partial x_i} \right| \leq \sum_{j=1}^n \frac{\partial^2 f(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)}{\partial x_i \partial x_j} \cdot \Delta x_i \cdot \Delta x_j.$$

This improves Eq. 1. In the report examples of this improvement will be given.

In this work, the foundations are presented for using fuzzy interval as an generic uncertainty characteristic for the results of computations with inaccurate initial data. It is worth noticing that it seems to be possible to extend this approach to other problems of metrology, beyond processing uncertain data: since this approach in a good agreement with generally accepted approaches in metrology and it can also use expert information.

References

1. N. Wiener.
2. L. Kantorovich. *In Russian*
3. R. Williamson and T. Downs. *International Journal of Approximate Reasoning*. **4**, 89 (1990).
4. D. Berleant and C. Goodman-Strauss. *Reliable Computing*. **4** (2), 147 (1998).
5. S. Ferson, V. Kreinovich, L. Ginzburg, D. Myers and K. Sentz. *Constructing probability boxes and Dempster-Shafer structures*, Technical report SAND2002-4015 (2003).
6. R. Moore. *Methods and Applications of Interval Analysis*. (1979).
7. L. de Figueiredo and J. Stolfi. *Numerical Algorithms*. **37** (1-4), 147 (2004).
8. M. Berz and G. Hoffstätter. *Reliable Computing*. **4** (1), 83 (1998).
9. L. Reznik, W. Jonson and G. Solopchenko. *Proc. NASA Conf. NAFIPS'94*. 405 (1994).
10. G. Corliss, C. Faure, A. Griewank, L. Hascoët and U. Naumann. *Automatic Differentiation of Algorithms: From Simulation to Optimization*. 383 (2002).
11. B. Hall. *Measurement Science and Technology*. **13** (4), 421 (2002).
12. L. Mari. *Measurement*. **42** (6), 844 (2009).
13. V. Kreinovich, C. Quintana and L. Reznik. *Gaussian membership functions are most adequate in representing uncertainty in measurements*. Technical Report, University of Texas at El-Paso (1992).
14. K. Semenov. *Informatics and its applications*. **6** (2), 101 (2012). In Russian.
15. V. Kreinovich, L. Reznik, K. Semenov and G. Solopchenko. *Proceedings of XX IMEKO World Congress*. paper IMEKO-WC-2012-ADC-O3 (2012).
16. H. Nguen, V. Kreinovich, C.-W. Tao, G. Solopchenko. *Soft Computing in Measurement and Information Acquisition*. 10 (2003).

17. P. Novitskiy and I. Zograf. *Errors estimation for measurements results*. (1991). In Russian.