If Many Physicists Are Right and No Physical Theory Is Perfect, Then by Using Physical Observations, We Can Feasibly Solve Almost All Instances of Each NP-Complete Problem

Olga Kosheleva[1], Michael Zakharevich[2], and
Vladik Kreinovich[1]
[1]University of Texas at El Paso
El Paso, Texas 79968, USA
olgak@utep.edu, vladik@utep.edu
[2]Aligh Technology Inc.
ymzakharevich@yahoo.com

Abstract

Many real-life problems are, in general, NP-complete, i.e., informally speaking, are difficult to solve – at least on computers based on the usual physical techniques. A natural question is: can the use of non-standard physics speed up the solution of these problems? This question has been analyzed for several specific physical theories, e.g., for quantum field theory, for cosmological solutions with wormholes and/or casual anomalies, etc. However, many physicists believe that no physical theory is perfect, i.e., that no matter how many observations support a physical theory, inevitably, new observations will come which will require this theory to be updated. In this paper, we show that if such a no-perfect-theory principle is true, then the use of physical data can drastically speed up the solution of NP-complete problems: namely, we can feasibly solve almost all instances of each NP-complete problem.

1 Formulation of the Problem

Solving NP-complete problems is important. In practice, we often need to find a solution that satisfies a given set of constraints – or at least check that such a solution is possible. Once we have a candidate for the solution, we can feasibly check whether this candidate indeed satisfies all the constraints. In theoretical computer science, "feasibly" is usually interpreted as computable in polynomial time, i.e., in time bounded by a polynomial of the length of the input.

A problem of checking whether a given set of constraints has a solution is called a problem of the class NP if we can check, in polynomial time, whether a given candidate is a solution; see, e.g., [11].

Examples of such problems includes checking whether a given graph can be colored in 3 colors, checking whether a given propositional formula – i.e., formula of the type

$$(v_1 \vee \neg v_2 \vee v_3) \,\&\, (v_4 \vee \neg v_2 \vee \neg v_5) \,\&\, \ldots,$$

is satisfiable, i.e., whether this formula is true by some combination of the propositional variables v_i, etc.

Each problem from the class NP can be algorithmically solved by trying all possible candidates. For example, we can check whether a graph can be colored by trying all possible assignments of colors to different vertices of a graph, and we can check whether a given propositional formula is satisfiable by trying all 2^n possible combinations of true-or-false values $v_1, \ldots, v_n$. Such exhaustive search algorithms require computation time like 2^n, time that grows exponentially with n. For medium-size inputs, e.g., for $n \approx 300$, the resulting time is larger than the lifetime of the Universe. So, these exhaustive search algorithms are not practically feasible.

It is not known whether problems from the class NP can be solved feasibly (i.e., in polynomial time): this is a famous open problem $\mathrm{P} \stackrel{?}{=} \mathrm{NP}$. It is known, however, there are problems in the class NP which are *NP-complete* in the sense that every problem from the class NP can be reduced to this problem. Reduction means, in particular, that if we can find a way to efficiently solve one NP-complete problem, then, by reducing other problems from the class NP to this problem, we can thus efficiently solve all the problems from the class NP.

So, it is very important to be able to efficiently solve even one NP-hard problem. (By the way, both above example of NP problems – checking whether a graph can be colored in 3 colors and whether a propositional formula is satisfiable – are NP-complete.)

Can the use of non-standard physics speed up the solution of NP-complete problems? NP-completeness of a problem means, crudely speaking, that the problem may take an unrealistically long time to solve – at least on computers based on the usual physical techniques. A natural question is: can the use of non-standard physics speed up the solution of these problems?

This question has been analyzed for several specific physical theories, e.g., for quantum filed theory, for cosmological solutions with wormholes and/or casual anomalies. Several possible techniques for solving NP-complete problems are described in [1, 6, 8, 9, 12].

No physical theory is perfect: a widely spread physicists' belief. If we prove that, within a given physical theory, we can speed up the solution to NP-complete problems, will this answer be fully satisfactory?

So far, in the history of physics, no matter how good a physical theory, no matter how good its accordance with observations, eventually, new observations appear which are not fully consistent with the original theory – and thus, a theory needs to be modified. For example, for several centuries, Newtonian physics seems to explain all observable facts – until later, quantum (and then relativistic) effects were discovered which required changes in physical theories.

Because of this history, many physicists believe that every physical theory is approximate – no matter how sophisticated a theory, no matter how accurate its current predictions, inevitably new observations will surface which would require a modification of this theory; see, e.g., [2].

How does this belief affect computations? At first glance, the fact that no theory is perfect makes the question of possible speed-up rather hopeless: no matter how good results we achieve within a given physical theory, eventually, this theory will turn out to be, strictly speaking, false – and thus, our speed-up scheme will not be applicable.

In this paper, we show, however, that in spite of this seeming hopelessness, an important speed-up can be deduced simply from the fact no physical theory is perfect.

Comment. A related question – whether we can use non-standard physical schemes to compute sequences which are not computable on traditional computational devices – was considered in [4, 7, 13].

2 How to Describe, in Precise Terms, that No Physical Theory Is Perfect

Discussion. The statement that no physical theory is perfect means that no matter what physical theory we have, eventually there will be observations which violate this theory. To formalize this statement, we need to formalize what are observations and what is a theory.

What are observations? Each observation can be represented, in the computer, as a sequence of 0s and 1s; actually, in many cases, the sensors already produce the signal in the computer-readable form, as a sequence of 0s and 1s.

An exact description of each experiment can also be described in precise terms, and thus, it will be represented in a computer as a sequence of 0s and 1s. An experiment should specify how long we wait for the result; in this way, we are guaranteed that we get the result.

In each experiment, we can specify which bit of the result we are interested in; for convenience, we can consider producing different bits as different experiments.

Each such experiment is represented as a sequence of 0s and 1s; by appending 1 at the beginning of this sequence, we can view this sequence as a binary expansion of a natural number i. This natural number will serve as the "code" describing the experiment. For example, a sequence 001 is transformed into $i = 1001_2 = 9_{10}$. (We need to append 1, because otherwise two different sequences 001 and 01 will be represented by the same integer).

For natural numbers i which correspond to experiment descriptions, let ω_i denote the bit result of the experiment described by the code i.

Let us also define ω_i for natural numbers i which do not correspond to a syntactically correct description of experiments. For example, we can fix a scheme of an experiment that uses a natural number i as a parameter (e.g., repeating a certain procedure i times), and define ω_i as the result of this scheme.

In these terms, all past and future observations form a (potentially) infinite sequence $\omega = \omega_1\omega_2\ldots$ of 0s and 1s, $\omega_i \in \{0, 1\}$.

What is a physical theory from the viewpoint of our problem: a set of sequences. A physical theory may be very complex, but all we care about is which sequences of observations ω are consistent with this theory and which are not. In other words, for our purposes, we can identify a physical theory T with the set of all sequences ω which are consistent with this theory.

Not every set of sequences corresponds to a physical theory: the set T must be non-empty and definable. Not every set of sequences comes from a physical theory. First, a physical theory must have at least one possible sequence of observations, i.e., the set T must be *non-empty.*

Second, a theory – and thus, the corresponding set – must be described by a finite sequence of symbols in an appropriate language. Sets which are uniquely by (finite) formulas are known as *definable.* Thus, the set T must be definable.

Since at any moment of time, we only have finitely many observations, the set T must be closed. Another property of a physical theory comes from the fact that at any given moment of time, we only have finitely many observations, i.e., we only observe finitely many bits. From this viewpoint, we say that observations $\omega_1\ldots\omega_n$ are consistent with the theory T if there is a continuing infinite sequence which is consistent with this theory, i.e., which belongs to the set T.

The only way to check whether an infinite sequence $\omega = \omega_1\omega_2\ldots$ is consistent with the theory is to check that for every n, the sequences $\omega_1\ldots\omega_n$ are consistent with the theory T. In other words, we require that for every infinite $\omega = \omega_1\omega_2\ldots$,

- if for every n, the sequence $\omega_1\ldots\omega_n$ is consistent with the theory T, i.e., if for every n, there exists a sequence $\omega^{(m)} \in T$ which has the same first n bits as ω, i.e., for which $\omega_i^{(m)} = \omega_i$ for all $i = 1, \ldots, n$,
- then the sequence ω itself should be consistent with the theory, i.e., this infinite sequence should also belong to the set T.

From the mathematical viewpoint, we can say that the sequences $\omega^{(m)}$ converge to ω: $\omega^{(m)} \to \omega$ (or, equivalently, $\lim \omega^{(m)} = \omega$), where convergence is understood in terms of the usual metric on the set of all infinite sequences $d(\omega, \omega') \stackrel{\text{def}}{=} 2^{-N(\omega,\omega')}$, where $N(\omega, \omega') \stackrel{\text{def}}{=} \max\{k : \omega_1\ldots\omega_k = \omega'_1\ldots\omega'_k\}$.

In general, if $\omega^{(m)} \to \omega$ in the sense of this metric, this means that for every n, there exists an integer ℓ such that for every $m \geq \ell$, we have $\omega_1^{(m)}\ldots\omega_n^{(m)} =$

$\omega_1 \ldots \omega_n$. Thus, if $\omega^{(m)} \in T$ for all m, this means that for every n, a finite sequence $\omega_1 \ldots \omega_n$ can be a part of an infinite sequence which is consistent with the theory T. In view of the above, this means that $\omega \in T$.

In other words, if $\omega^{(m)} \to \omega$ and $\omega^{(m)} \in T$ for all m, then $\omega \in T$. So, the set T must contain all the limits of all its sequences. In topological terms, this means that the set T must be *closed*.

A physical theory must be different from a fact and hence, the set T must be nowhere dense. The assumption that we are trying to formalize is that no matter how many observations we have which confirm a theory, there eventually will be a new observation which is inconsistent with this theory. In other words, for every finite sequence $\omega_1 \ldots \omega_m$ which is consistent with the set T, there exists a continuation of this sequence which does not belong to T. The opposite would be if all the sequences which start with $\omega_1 \ldots \omega_m$ belong to T; in this case, the set T will be *dense* in the open set of all the sequences starting with $\omega_1 \ldots \omega_m$. Thus, in mathematical terms, the statement that every finite sequence which is consistent with T has a continuation which is not consistent with T means that the set T is *nowhere dense.*

Resulting definition of a theory. By combining the above properties of a set T which describes a physical theory, we arrive at the following definition.

Definition 1. *By a* physical theory*, we mean a non-empty closed nowhere dense definable set T.*

Mathematical comment. To properly define what is definable, we need to have a consistent formal definition of definability. In this paper, we follow a natural definition from [5, 6, 7, 13] – which is reproduced in the Appendix.

Formalization of the principle that no physical theory is perfect. In terms of the above notations, the no-perfect-theory principle simply means that the infinite sequence ω (describing the actual results of all observations) is not consistent with any physical theory, i.e., that the sequence ω does not belong to any physical theory T. Thus, we arrive at the following definition.

Definition 2. *We say that an infinite binary sequence ω is* consistent with the no-perfect-theory principle *if the sequence ω does not belong to any physical theory (in the sense of Definition 1).*

Comment. Are there such sequences in the first place? Our answer is yes. Indeed, by definition, we want a sequence which does not belong to a union of all definable physical theories. Every physical theory is closed nowhere dense set. Every definable set is defined by a finite sequence of symbols, so there are no more than countably many definable theories. Thus, the union of all definable physical theories is contained in a union of countably many closed nowhere dense sets. Such sets are knows as *meager* (or *Baire first category*); it is known that the set of all infinite binary sequences is not meager. Thus, there are sequences who do not belong to the above union – i.e., sequences which are consistent with the no-perfect-theory principle; see, e.g., [3, 10].

3 Main Result: The Use of Physical Observations Can Help in Solving NP-Complete Problems

What we do in this section. In this section, we prove that under the no-perfect-theory principle, it is possible to drastically speed up the solution of NP-complete problems.

How to represent instances of an NP-complete problem. For each NP-complete problem $\mathcal{P}$, its instances are sequences of symbols. In the computer, each such sequence is represented as a sequence of 0s and 1s. Thus, as in the previous section, we can append 1 in front of this sequence and interpret the resulting sequence as a binary code of a natural number i.

In principle, not all natural numbers i correspond to instances of a problem $\mathcal{P}$; we will denote the set of all natural numbers which correspond to such instances by $S_{\mathcal{P}}$.

For each $i \in S_{\mathcal{P}}$, the correct answer (true or false) to the i-th instance of the problem $\mathcal{P}$ will be denoted by $s_{\mathcal{P},i}$.

Easier-to-solve and harder-to-solve NP-complete problems. We will show that our method works on "harder-to-solve" NP-complete problems, harder-to-solve in the following sense. By definition, for all NP-complete problems, unless P = NP, there is no feasible algorithm for solving all its instances. However, for some easier-to-solve problems, there are feasible algorithms which solve "almost all" instances, in the sense that for each n, the proportional of instance $i \leq n$ for which the problem is solved by this algorithm tends to 1. In this case, while the worst-case complexity is still exponential, in practice, almost all problems can be feasibly solved.

A more challenging case are harder-to-solve NP-complete problems, for which no feasible algorithm is known that would solve almost all instances.

In this section, we show that our method works on all NP-complete problems, both easier-to-solve and harder-to-solve ones.

What we mean by using physical observations in computations. In addition to performing computations, our computational device can produce a scheme i for an experiment, and then use the result ω_i of this experiment in future computations. In other words, given an integer i, we can produce ω_i.

In precise theory-of-computation terms, the use of physical observations in computations thus means computations that use the sequence ω as an *oracle*; see, e.g., [11].

Definition 3. *By a* ph-algorithm $\mathcal{A}$*, we mean an algorithm which uses, as an oracle, a sequence* ω *which is consistent with the no-perfect-theory principle.*

Notation. The result of applying an algorithm $\mathcal{A}$ using ω to an input i will be denoted by $\mathcal{A}(\omega, i)$.

Definition 4. *Let* $\mathcal{P}$ *be an NP-complete problem. We say that a feasible ph-algorithm* $\mathcal{A}$ solves almost all instances of $\mathcal{P}$ *if for every* $\varepsilon > 0$*, and for every* *natural number n, there exists an integer $N \ge n$ for which the proportion of the instances $i \le N$ of the problem $\mathcal{P}$ which are correctly solved by $\mathcal{A}$ is greater than $1-\varepsilon$:*

$$\forall \varepsilon > 0\, \forall n\, \exists N \left(N \ge n \,\&\, \frac{\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, \mathcal{A}(\omega, i) = s_{\mathcal{P},i}\}}{\#\{i \le N : i \in S_{\mathcal{P}}\}} > 1 - \varepsilon \right).$$

Comment. The restriction to sufficiently long inputs $N \ge n$ makes perfect sense: for short inputs, NP-completeness is not an issue: we can perform exhaustive search of all possible bit sequences of length 10, 20, and even 30. The challenge starts when the length of the input is high.

Proposition 1. *For every NP-complete problem $\mathcal{P}$, there exists a feasible ph-algorithm $\mathcal{A}$ that solves almost all instances of $\mathcal{P}$.*

Comments. In other words, we show that the use of physical observations makes all NP-complete problems easier-to-solve (in the above-described sense).

It turns out that this result is the best possible, in the sense that the use of physical observations cannot solve *all* instances.

Proposition 2. *If P≠NP, then no feasible ph-algorithm $\mathcal{A}$ can solve all instances of $\mathcal{P}$.*

Comment. Another possible idea of strengthening Proposition 1 is to require that the property

$$\frac{\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, \mathcal{A}(\omega, i) = s_{\mathcal{P},i}\}}{\#\{i \le N : i \in S_{\mathcal{P}}\}} > 1 - \varepsilon$$

hold not only for *infinitely many* N, but for *all* N starting with some N_0. It turns out that in this formulation, the use of physical observation does not help.

Definition 5. *Let $\mathcal{P}$ be an NP-complete problem. Let $\delta > 0$ be a real number. We say that a feasible ph-algorithm $\mathcal{A}$ δ-*solves* $\mathcal{P}$ if*

$$\exists N_0\, \forall N \left(N \ge N_0 \to \frac{\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, \mathcal{A}(\omega, i) = s_{\mathcal{P},i}\}}{\#\{i \le N : i \in S_{\mathcal{P}}\}} > \delta \right).$$

Proposition 3. *For every NP-complete problem $\mathcal{P}$ and for every $\delta > 0$, if there exists a feasible ph-algorithm $\mathcal{A}$ that δ-solves $\mathcal{P}$, then there exists a feasible algorithm $\mathcal{A}'$ (not using physical observations) which also δ-solves $\mathcal{P}$.*

4 Proofs

Proof of Proposition 1.

1°. As the desired ph-algorithm, we will, given an instance i, simply produce the result ω_i of the i-th experiment. Let us prove, by contradiction, that this algorithm satisfies the desired property.

2°. We want to prove that for every $\varepsilon > 0$ and for every n, there exists an integer $N \geq n$ for which

$$\#\{i \leq N : i \in S_{\mathcal{P}} \,\&\, \omega_i = s_{\mathcal{P},i}\} > (1-\varepsilon) \cdot \#\{i \leq N : i \in S_{\mathcal{P}}\}.$$

The assumption that this property is not satisfied means that for some $\varepsilon > 0$ and for some integer n, we have

$$\#\{i \leq N : i \in S_{\mathcal{P}} \,\&\, \omega_i = s_{\mathcal{P},i}\} \leq (1-\varepsilon) \cdot \#\{i \leq N : i \in S_{\mathcal{P}}\} \text{ for all } N \geq n. \quad (1)$$

Let T denote the set of all the sequences x that satisfy the property (1), i.e., let

$$T \stackrel{\text{def}}{=} \{x : \#\{i \leq N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} \leq$$
$$(1-\varepsilon) \cdot \#\{i \leq N : i \in S_{\mathcal{P}}\} \text{ for all } N \geq n\}.$$

We will prove that this set T is a physical theory in the sense of Definition 1.

Then, due to Definition 2 and the fact that the sequence ω satisfies the no-perfect-theory principle, we will be able to conclude that $\omega \notin T$, and thus, that the property (1) is not satisfied for the given sequence ω. This will conclude the proof by contradiction.

3°. By definition of a physical theory T, it is a set which is non-empty, closed, nowhere dense, and definable. Let us prove these four properties one by one.

3.1°. Non-emptiness comes from the fact that the sequence x_i for which $x_i = \neg s_{\mathcal{P},i}$ for $i \in S_{\mathcal{P}}$ and $x_i = 0$ otherwise clearly belongs to this set: for this sequence, for every N, we have

$$\#\{i \leq N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} = 0$$

and thus, the desired property is satisfied.

3.2°. Let us prove that the set T is closed, i.e., that if we have a family of sequences $x^{(m)} \in T$ for which $x^{(m)} \to \omega$, then $x \in T$.

Indeed, let us take any $N \neq n$, and let us prove that

$$\#\{i \leq N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} \leq (1-\varepsilon) \cdot \#\{i \leq N : i \in S_{\mathcal{P}}\}$$

for this N. Due to $x^{(m)} \to x$, there exists M for which, for all $m \geq M$, the first N bits of $x^{(m)}$ coincide with the first N bits of the sequence x: $x_i^{(m)} = \omega_i$ for all $i \leq N$. Thus,

$$\#\{i \leq N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} = \#\{i \leq N : i \in S_{\mathcal{P}} \,\&\, x_i^{(m)} = s_{\mathcal{P},i}\}.$$

Since $x^{(m)} \in T$, we have

$$\#\{i \leq N : i \in S_{\mathcal{P}} \,\&\, x_i^{(m)} = s_{\mathcal{P},i}\} \leq (1-\varepsilon) \cdot \#\{i \leq N : i \in S_{\mathcal{P}}\},$$

thus

$$\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} \le (1-\varepsilon) \cdot \#\{i \le N : i \in S_{\mathcal{P}}\}.$$

So, the set T is indeed closed.

3.3°. Let us now prove that the set T is nowhere dense, i.e., that for every finite sequence $x_1 \ldots x_m$, there exists a continuation x which does not belong to the set T.

Indeed, as such a continuation, we can simply take a sequence

$$x = x_1 \ldots x_m x_{m+1} x_{m+2} \ldots$$

where for $i > m$, we take $x_i = s_{\mathcal{P},i}$ if $i \in S_{\mathcal{P}}$ and $x_i = 0$ otherwise. For this new sequence, for every N, at most m first instances may lead to results different from $s_{\mathcal{P},i}$, so we have

$$\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} \ge \#\{i \le N : i \in S_{\mathcal{P}}\} - m.$$

When $N \to \infty$, then $\#\{i \le N : i \in S_{\mathcal{P}}\} \to \infty$, so for sufficiently large N, we have

$$\#\{i \le N : i \in S_{\mathcal{P}}\} - m > (1-\varepsilon) \cdot \#\{i \le N : i \in S_{\mathcal{P}}\},$$

thus,

$$\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} > (1-\varepsilon) \cdot \#\{i \le N : i \in S_{\mathcal{P}}\},$$

and we cannot have

$$\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, x_i = s_{\mathcal{P},i}\} \le (1-\varepsilon) \cdot \#\{i \le N : i \in S_{\mathcal{P}}\}.$$

Therefore, this continuation does not belong to the set T.

3.4°. Finally, since the formula (1) explicitly defines the set T, this set T is clearly definable.

So, T is a physical theory, hence $\omega \notin T$, and the proposition is proven.

Proof of Proposition 2. Let us assume that P≠NP. We then need to prove that for every feasible ph-algorithm $\mathcal{A}$, it is not possible to have

$$\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, \mathcal{A}(\omega, i) = s_{\mathcal{P},i}\} = \#\{i \le N : i \in S_{\mathcal{P}}\}$$

for all natural numbers N.

To prove this impossibility, let us consider, for each feasible ph-algorithm $\mathcal{A}$, the set

$$T(\mathcal{A}) \stackrel{\text{def}}{=} \{x : \#\{i \le N : i \in S_{\mathcal{P}} \,\&\, \mathcal{A}(x, i) = s_{\mathcal{P},i}\} = \\ \#\{i \le N : i \in S_{\mathcal{P}}\} \text{ for all } N\}.$$

Similarly to the proof of Proposition 1, we can show that this set $T(\mathcal{A})$ is closed and definable.

Let us prove that the set $T(\mathcal{A})$ is nowhere dense, i.e., that for every finite sequence $x_1 \ldots x_m$, there exists a continuation x which does not belong to the set $T(\mathcal{A})$. Indeed, we can simply extend the original finite sequence $x_1 \ldots x_m$ by 0s. In this case, when the oracle has only finitely many nonzero bits, we can incorporate these bits into an algorithm and get a feasible non-oracle algorithm $\mathcal{A}'$ which produces the same results: $\mathcal{A}'(i) = \mathcal{A}(x, i)$ for all i.

Let us prove, by contradiction, that $x \notin T(\mathcal{A})$. Indeed, if $x \in T(\mathcal{A})$, this would mean that

$$\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, \mathcal{A}'(i) = s_{\mathcal{P},i}\} = \#\{i \le N : i \in S_{\mathcal{P}}\}$$

for all N. Thus, the feasible non-oracle algorithm $\mathcal{A}'$ solves all the instances of the original NP-complete problem $\mathcal{P}$, which contradicts to our assumption that P$\neq$NP. This contradiction proves that $x \notin T(\mathcal{A})$ and thus, the set $T(\mathcal{A})$ is indeed nowhere dense.

We have thus proven that the set $T(\mathcal{A})$ is closed, nowhere dense, and definable. The only property which is still missing from the definition of a physical theory (Definition 1) is non-emptiness. We do not know whether the set $T(\mathcal{A})$ is non-empty or not, but we can prove the desired impossibility in both cases.

If the set $T(\mathcal{A})$ is non-empty, then this set is a theory in the sense of Definition 1, and thus, since the sequence ω satisfies the no-perfect-theory principle, we have $\omega \notin T(\mathcal{A})$. This means that the ph-algorithm $\mathcal{A}$ is not solving all instances of the problem $\mathcal{P}$.

If the set $T(\mathcal{A})$ is empty, this also means that the ph-algorithm $\mathcal{A}$ does not solve all instances of the problem $\mathcal{P}$ – no matter what oracle we use.

The proposition is proven.

Proof of Proposition 3. Let us assume that no non-oracle feasible algorithm δ-solves the problem $\mathcal{P}$. We then need to prove that for every feasible ph-algorithm $\mathcal{A}$, it is not possible to have N_0 for which

$$\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, \mathcal{A}(\omega, i) = s_{\mathcal{P},i}\} > \delta \cdot \#\{i \le N : i \in S_{\mathcal{P}}\}$$

for all natural numbers $N \ge N_0$.

To prove this impossibility, let us consider, for each feasible ph-algorithm $\mathcal{A}$ and for each natural number N_0, the set

$$T(\mathcal{A}, N_0) \stackrel{\text{def}}{=} \{x : \#\{i \le N : i \in S_{\mathcal{P}} \,\&\, \mathcal{A}(x, i) = s_{\mathcal{P},i}\} >$$

$$\delta \cdot \#\{i \le N : i \in S_{\mathcal{P}}\} \text{ for all } N \ge N_0\}.$$

Similarly to the proof of Proposition 1, we can show that this set $T(\mathcal{A}, N_0)$ is closed and definable.

Let us prove that the set $T(\mathcal{A}, N_0)$ is nowhere dense, i.e., that for every finite sequence $x_1 \ldots x_m$, there exists a continuation x which does not belong to the set $T(\mathcal{A}, N_0)$. Indeed, similarly to the proof of Proposition 2, we can extend the original finite sequence $x_1 \ldots x_m$ by 0s. In this case, when the oracle has only finitely many nonzero bits, we can incorporate these bits into an algorithm and get a feasible non-oracle algorithm $\mathcal{A}'$ which produces the same results: $\mathcal{A}'(i) = \mathcal{A}(x, i)$ for all i.

Let us prove, by contradiction, that $x \notin T(\mathcal{A}, N_0)$. Indeed, if $x \in T(\mathcal{A}, N_0)$, this would mean that

$$\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, \mathcal{A}'(i) = s_{\mathcal{P},i}\} > \delta \cdot \#\{i \le N : i \in S_{\mathcal{P}}\}$$

for all $N \ge N_0$. Thus, the feasible non-oracle algorithm $\mathcal{A}'$ δ-solves the original NP-complete problem $\mathcal{P}$, which contradicts to our assumption that no such feasible non-oracle algorithm is possible. This contradiction proves that $x \notin T(\mathcal{A}, N_0)$ and thus, the set $T(\mathcal{A}, N_0)$ is indeed nowhere dense.

We have thus proven that the set $T(\mathcal{A}, N_0)$ is closed, nowhere dense, and definable. The only property which is still missing from the definition of a physical theory (Definition 1) is non-emptiness. We do not know whether the set $T(\mathcal{A}, N_0)$ is non-empty or not, but we can prove the desired impossibility in both cases.

For each N_0, if the set $T(\mathcal{A}, N_0)$ is non-empty, then this set is a theory in the sense of Definition 1, and thus, since the sequence ω satisfies the no-perfect-theory principle, we have $\omega \notin T(\mathcal{A}, N_0)$, i.e.,

$$\#\{i \le N : i \in S_{\mathcal{P}} \,\&\, \mathcal{A}(\omega, i) = s_{\mathcal{P},i}\} \le$$
$$\delta \cdot \#\{i \le N : i \in S_{\mathcal{P}}\} \text{ for some } N \ge N_0. \tag{2}$$

If the set $T(\mathcal{A}, N_0)$ corresponding to a given N_0 is empty, then also $\omega \notin T(\mathcal{A}, N_0)$, i.e., we also have the property (2).

Since the property (2) holds for all N_0, this means that the ph-algorithm $\mathcal{A}$ does not δ-solve the problem $\mathcal{P}$.

The proposition is proven.

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A Appendix: A Formal Definition of Definable Sets

Definition A1. *Let $\mathcal{L}$ be a theory, and let $P(x)$ be a formula from the language of the theory $\mathcal{L}$, with one free variable x for which the set $\{x \mid P(x)\}$ is defined in the theory $\mathcal{L}$. We will then call the set $\{x \mid P(x)\}$ $\mathcal{L}$-*definable*.*

Crudely speaking, a set is $\mathcal{L}$-definable if we can explicitly *define* it in $\mathcal{L}$. The set of all real numbers, the set of all solutions of a well-defined equation, every set that we can describe in mathematical terms: all these sets are $\mathcal{L}$-definable.

This does not mean, however, that *every* set is $\mathcal{L}$-definable: indeed, every $\mathcal{L}$-definable set is uniquely determined by formula $P(x)$, i.e., by a text in the language of set theory. There are only denumerably many words and therefore, there are only denumerably many $\mathcal{L}$-definable sets. Since, e.g., in a standard model of set theory ZF, there are more than denumerably many sets of integers, some of them are thus not $\mathcal{L}$-definable.

Our objective is to be able to make mathematical statements about $\mathcal{L}$-definable sets. Therefore, in addition to the theory $\mathcal{L}$, we must have a stronger theory $\mathcal{M}$ in which the class of all $\mathcal{L}$-definable sets is a set – and it is a countable set.

Denotation. *For every formula F from the theory $\mathcal{L}$, we denote its Gödel number by $\lfloor F \rfloor$.*

Comment. A Gödel number of a formula is an integer that uniquely determines this formula. For example, we can define a Gödel number by describing what this formula will look like in a computer. Specifically, we write this formula in LaTeX, interpret every LaTeX symbol as its ASCII code (as computers do), add 1 at the beginning of the resulting sequence of 0s and 1s, and interpret the resulting binary sequence as an integer in binary code.

Definition A2. *We say that a theory $\mathcal{M}$ is* stronger *than $\mathcal{L}$ if it contains all formulas, all axioms, and all deduction rules from $\mathcal{L}$, and also contains a special predicate* $\mathrm{def}(n, x)$ *such that for every formula $P(x)$ from $\mathcal{L}$ with one free variable, the formula*

$$\forall y\,(\mathrm{def}(\lfloor P(x) \rfloor, y) \leftrightarrow P(y))$$

is provable in $\mathcal{M}$.

The existence of a stronger theory can be easily proven: indeed, for $\mathcal{L}$=ZF, there exists a stronger theory $\mathcal{M}$. As an example of such a stronger theory, we can simply take the theory $\mathcal{L}$ plus all countably many equivalence formulas as described in Definition A2 (formulas corresponding to all possible formulas $P(x)$ with one free variable). This theory clearly contains $\mathcal{L}$ and all the desired equivalence formulas, so all we need to prove is that the resulting theory $\mathcal{M}$

is consistent (provided that $\mathcal{L}$ is consistent, of course). Due to compactness principle, it is sufficient to prove that for an arbitrary finite set of formulas $P_1(x), \ldots, P_m(x)$, the theory $\mathcal{L}$ is consistent with the above reflection-principle-type formulas corresponding to these properties $P_1(x), \ldots, P_m(x)$.

This auxiliary consistency follows from the fact that for such a finite set, we can take

$$\mathrm{def}(n, y) \leftrightarrow (n = \lfloor P_1(x) \rfloor \,\&\, P_1(y)) \vee \ldots \vee (n = \lfloor P_m(x) \rfloor \,\&\, P_m(y)).$$

This formula is definable in $\mathcal{L}$ and satisfies all m equivalence properties. The statement is proven.

Important comments. In the main text, we will assume that a theory $\mathcal{M}$ that is stronger than $\mathcal{L}$ has been fixed; proofs will mean proofs in this selected theory $\mathcal{M}$.

An important feature of a stronger theory $\mathcal{M}$ is that the notion of an $\mathcal{L}$-definable set can be expressed within the theory $\mathcal{M}$: a set S is $\mathcal{L}$-definable if and only if

$$\exists n \in \mathbb{N}\, \forall y (\mathrm{def}(n, y) \leftrightarrow y \in S).$$

In the paper, when we talk about definability, we will mean this property expressed in the theory $\mathcal{M}$. So, all the statements involving definability become statements from the theory $\mathcal{M}$ itself, *not* statements from metalanguage.