# How Much For an Interval? a Set? a Twin Set? a p-Box? A Kaucher Interval? Towards an Economics-Motivated Approach to Decision Making Under Uncertainty

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**Abstract.** A natural idea of decision making under uncertainty is to assign a fair price to different alternatives, and then to use these fair prices to select the best alternative. In this paper, we show how to assign a fair price under different types of uncertainty.

**Keywords:** decision making, interval uncertainty, set uncertainty, p-box

## 1 Decision Making under Uncertainty: Formulation of the Problem

In many practical situations, we have several alternatives, and we need to select one of these alternatives. For example:

- a person saving for retirement needs to find the best way to invest money;
- a company needs to select a location for its new plant;
- a designer must select one of several possible designs for a new airplane;
- a medical doctor needs to select a treatment for a patient, etc.

Decision making is the easiest if we know the exact consequences of selecting each alternative. Often, however, we only have an incomplete information about consequences of different alternative, and we need to select an alternative under this uncertainty.

Traditional decision theory (see, e.g., [8, 12]) assumes that for each alternative $a$, we know the probability $p_i(a)$ of different outcomes $i$. It can be proven that preferences of a rational decision maker can be described by *utilities* $u_i$ so that an alternative $a$ is better if its expected utility $u(a) \stackrel{\text{def}}{=} \sum_i p_i(a) \cdot u_i$ is larger.

Often, we do not know the probabilities $p_i(a)$. As a result, we do not know the exact value of the gain $u$ corresponding to each alternative. How can we then make a decision?

For the case when we only know the interval $[\underline{u}, \overline{u}]$ containing the actual (unknown) value of the gain $u$, a possible solution was proposed in the 1950s by a future Nobelist L. Hurwicz [5, 8]: we should select an alternative that maximizes the value $\alpha_H \cdot \overline{u}(a) + (1 - \alpha_H) \cdot \underline{u}(a)$. Here, the parameter $\alpha_H \in [0, 1]$ described the optimism level of a decision maker:

- $\alpha_H = 1$ means optimism;
- $\alpha_H = 0$ means pessimism;
- $0 < \alpha_H < 1$ combines optimism and pessimism.

Hurwicz's approach is widely used in decision making, but it is largely a heuristic, and it is not clear how to extend it other types of uncertainty. It is therefore desirable to develop more theoretically justified recommendations for decision making under uncertainty, recommendations that would be applicable to different types of uncertainty.

In this paper, we propose such recommendations by explaining how to assign a fair price to each alternative, so that we can select between several alternatives by comparing their fair prices.

The structure of this paper is as follows: in Section 2, we recall how to describe different types of uncertainty; in Section 3, we describe the fair price approach; in the following sections, we show how the fair price approach can be applied to different types of uncertainty.

*Comment.* Our result for the case of interval uncertainty has been previously described in [9]; other results are new.

## 2 How to Describe Uncertainty

When we have a full information about a situation, then we can express our desirability of each possible alternative by declaring a price that we are willing to pay for this alternative. Once these prices are determined, we simply select the alternative for which the corresponding price is the highest. In this full information case, we know the exact gain $u$ of selecting each alternative.

In practice, we usually only have partial information about the gain $u$: based on the available information, there are several possible values of the gain $u$. In other words, instead of the exact gain $u$, we only know a *set* $S$ of possible values of the gain.

We usually know lower and bounds for this set, so this set is *bounded*. It is also reasonable to assume that the set $S$ is *closed*: indeed, if we have a sequence of possible values $u_n \in S$ that converges to a number $u_0$, then, no matter how accurately we measure the gain, we can never distinguish between the limit value $u_0$ and a sufficiently close value $u_n$. Thus, we will never be able to conclude that the limit value $u_0$ is not possible – and thus, it is reasonable to consider it possible, i.e., to include the limit point $u_0$ into the set $S$ of possible values.

In many practical situations, if two gain values $u < u'$ are possible, then all intermediate values $u'' \in (u, u')$ are possible as well. In this case, the bounded closed set $S$ is simply an *interval* $[\underline{u}, \overline{u}]$.

However, sometimes, some intermediate numbers $u''$ cannot be possible values of the gain. For example, if we buy an obscure lottery ticket for a simple prize-or-no-prize lottery from a remote country, we either get the prize or lose the money. In this case, the set of possible values of the gain consists of two values. To account for such situations, we need to consider general bounded closed sets.

In addition to knowing which gain values are possible, we may also have an information about which of these values are more probable and which values are less probable. Sometimes, this information has a *qualitative* nature, in the sense that, in addition to the set $S$ of possible gain values, we also know a (closed) subset $s \subseteq S$ of values which are more probable (so that all the values from the difference $S - s$ are less probable). In many cases, the set $s$ also contains all its intermediate values, so it is an interval; an important particular case is when this interval $s$ consists of a single point. In other cases, the set $s$ may be different from an interval.

Often, we have a *quantitative* information about the probability (frequency) of different values $u \in S$. A universal way to describe a probability distribution on the real line is to describe its cumulative distribution function (cdf) $F(u) \stackrel{\text{def}}{=} \text{Prob}(U \leq u)$. In the ideal case, we know the exact cdf $F(u)$. In practice, we usually only know the values of the cdf with uncertainty. Typically, for every $u$, we may only know the bounds $\underline{F}(u)$ and $\overline{F}(u)$ on the actual (unknown) values $F(u)$. The corresponding interval-valued function $[\underline{F}(u), \overline{F}(u)]$ is known as a *p-box* [2, 3].

All this classification relates to the usual *passive* uncertainty, uncertainty over which we have no control. Sometimes, however, we have *active* uncertainty. As an example, let us consider two situations in which we need to minimize the amount of energy $E$ used to heat the building. For simplicity, let us assume that cooling by 1 degree requires 1 unit of energy.

In the first situation, we simply know the interval $[\underline{E}, \overline{E}]$ that contains the actual (unknown) value of the energy $E$: for example, we know that $E \in [20, 25]$ (and we do not control this energy). In the second situation, we know that the outside temperature is between 50 F and 55 F, and we want to maintain the temperature 75 F. In this case, we also conclude that $E \in [20, 25]$, but this time, we ourselves (or, alternatively, the heating system programmed by us) set up the appropriate amount of energy.

The distinction between the usual (passive) uncertainty and a different (active) type of uncertainty can be captured by considering *improper intervals* first introduced by Kaucher, i.e., intervals $[\underline{u}, \overline{u}]$ in which we may have $\underline{u} > \overline{u}$ see, e.g., [7, 13]. For example, in terms of these Kaucher intervals, our first (passive) situation is described by the interval $[15, 20]$, while the second (active) situation is described by an improper interval $[20, 15]$.

In line with this classification of different types of uncertainty, in the following text, we will first consider the simplest (interval) uncertainty, then the general set-valued uncertainty, then uncertainty described by a pair of embedded sets (in particular, by a pair of embedded intervals). After that, we consider situations with known probability distribution, situations with a known p-box, and finally, situations described by Kaucher intervals.

## 3 Fair Price Approach: Main Idea

When we have full information, we can express our desirability of each possible situation by declaring a price that we are willing to pay to get involved in this situation. To make decisions under uncertainty, it is therefore desirable to assign a fair price to each uncertain situation: e.g., to assign a fair price to each interval and/or to each set.

There are reasonable restrictions on the function that assigns the fair price to each type of uncertainty. First, the fair price should be *conservative*: if we know that the gain is always larger than or equal to $\underline{u}$, then the fair price corresponding to this situation should also be greater than or equal to $\underline{u}$. Similarly, if we know that the gain is always smaller than or equal to $\overline{u}$, then the fair price corresponding to this situation should also be smaller than or equal to $\overline{u}$.

Another natural property is *monotonicity:* if one alternative is clearly better than the other, then its fair price should be higher (or at least not lower).

Finally, the fair price should be *additive* in the following sense. Let us consider the situation when we have two consequent independent decisions. In this case, we can either consider two decision processes separately, or we can consider a single decision process in which we select a pair of alternatives:

- the 1st alternative corresponding to the 1st decision, and
- the 2nd alternative corresponding to the 2nd decision.

If we are willing to pay the amount $u$ to participate in the first process, and we are willing to pay the amount $v$ to participate in the second decision process, then it is reasonable to require that we should be willing to pay $u + v$ to participate in both decision processes.

On the examples of the above-mentioned types of uncertainty, let us describe the formulas for the fair price that can be derived from these requirements.

## 4 Case of Interval Uncertainty

We want to assign, to each interval $[\underline{u}, \overline{u}]$, a number $P([\underline{u}, \overline{u}])$ describing the fair price of this interval. Conservativeness means that the fair price $P([\underline{u}, \overline{u}])$ should be larger than or equal to $\underline{u}$ and smaller than or equal to $\overline{u}$, i.e., that the fair price of an interval should be located in this interval:

$$P([\underline{u}, \overline{u}]) \in [\underline{u}, \overline{u}].$$

Let us now apply monotonicity. Suppose that we keep the lower endpoint $\underline{u}$ intact but increase the upper bound. This means that we keep all the previous possibilities, but we also add new possibilities, with a higher gain. In other words, we are improving the situation. In this case, it is reasonable to require that after this addition, the fair price should either increase or remain the same, but it should definitely not decrease:

$$\text{if } \underline{u} = \underline{v} \text{ and } \overline{u} < \overline{v} \text{ then } P([\underline{u}, \overline{u}]) \le P([\underline{v}, \overline{v}]).$$

Similarly, if we dismiss some low-gain alternatives, this should increase (or at least not decrease) the fair price:

$$\text{if } \underline{u} < \underline{v} \text{ and } \overline{u} = \overline{v} \text{ then } P([\underline{u}, \overline{u}]) \le P([\underline{v}, \overline{v}]).$$

Finally, let us apply additivity. In the case of interval uncertainty, about the gain $u$ from the first alternative, we only know that this (unknown) gain is in $[\underline{u}, \overline{u}]$. Similarly, about the gain $v$ from the second alternative, we only know that this gain belongs to the interval $[\underline{v}, \overline{v}]$.

The overall gain $u + v$ can thus take any value from the interval

$$[\underline{u}, \overline{u}] + [\underline{v}, \overline{v}] \stackrel{\text{def}}{=} \{u + v : u \in [\underline{u}, \overline{u}], v \in [\underline{v}, \overline{v}]\}.$$

It is easy to check that (see, e.g., [6, 10]):

$$[\underline{u}, \overline{u}] + [\underline{v}, \overline{v}] = [\underline{u} + \underline{v}, \overline{u} + \overline{v}].$$

Thus, for the case of interval uncertainty, the additivity requirement about the fair prices takes the form

$$P([\underline{u} + \underline{v}, \overline{u} + \overline{v}]) = P([\underline{u}, \overline{u}]) + P([\underline{v}, \overline{v}]).$$

So, we arrive at the following definition:

**Definition 1.** *By a* fair price under interval uncertainty*, we mean a function* $P([\underline{u}, \overline{u}])$ *for which:*

- $\underline{u} \le P([\underline{u}, \overline{u}]) \le \overline{u}$ *for all* $\underline{u}$ *and* $\overline{u}$ *(conservativeness);*
- *if* $\underline{u} = \underline{v}$ *and* $\overline{u} < \overline{v}$*, then* $P([\underline{u}, \overline{u}]) \le P([\underline{v}, \overline{v}])$ *(monotonicity);*
- *(additivity) for all* $\underline{u}$*,* $\overline{u}$*,* $\underline{v}$*, and* $\overline{v}$*, we have*

$$P([\underline{u} + \underline{v}, \overline{u} + \overline{v}]) = P([\underline{u}, \overline{u}]) + P([\underline{v}, \overline{v}]).$$

**Proposition 1.** [9] *Each fair price under interval uncertainty has the form*

$$P([\underline{u}, \overline{u}]) = \alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u} \text{ for some } \alpha_H \in [0, 1].$$

*Comment.* We thus get a new justification of Hurwicz optimism-pessimism criterion.

**Proof.**

1°. Due to monotonicity, $P([u, u]) = u$.

2°. Also, due to monotonicity, $\alpha_H \stackrel{\text{def}}{=} P([0, 1]) \in [0, 1]$.

3°. For $[0, 1] = [0, 1/n] + \ldots + [0, 1/n]$ ($n$ times), additivity implies $\alpha_H = n \cdot P([0, 1/n])$, so $P([0, 1/n]) = \alpha_H \cdot (1/n)$.

4°. For $[0, m/n] = [0, 1/n] + \ldots + [0, 1/n]$ ($m$ times), additivity implies

$$P([0, m/n]) = \alpha_H \cdot (m/n).$$

5°. For each real number $r$, for each $n$, there is an $m$ such that $m/n \le r \le (m+1)/n$. Monotonicity implies

$$\alpha_H \cdot (m/n) = P([0, m/n]) \le P([0, r]) \le P([0, (m+1)/n]) = \alpha_H \cdot ((m+1)/n).$$

When $n \to \infty$, $\alpha_H \cdot (m/n) \to \alpha_H \cdot r$ and $\alpha_H \cdot ((m+1)/n) \to \alpha_H \cdot r$, hence $P([0, r]) = \alpha_H \cdot r$.

6°. For $[\underline{u}, \overline{u}] = [\underline{u}, \underline{u}] + [0, \overline{u} - \underline{u}]$, additivity implies $P([\underline{u}, \overline{u}]) = \underline{u} + \alpha_H \cdot (\overline{u} - \underline{u})$. The proposition is proven.

## 5 Case of Set-Valued Uncertainty

Intervals are a specific case of bounded closed sets. We already know how to assign fair price to intervals. So, we arrive at the following definition.

**Definition 2.** *By a fair price under set-valued uncertainty, we mean a function $P$ that assigns, to every bounded closed set $S$, a real number $P(S)$, for which:*

- $P([\underline{u}, \overline{u}]) = \alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u}$ *(conservativeness);*
- $P(S + S') = P(S) + P(S')$, *where* $S + S' \stackrel{\text{def}}{=} \{s + s' : s \in S, s' \in S'\}$ *(additivity).*

**Proposition 2.** *Each fair price under set uncertainty has the form $P(S) = \alpha_H \cdot \sup S + (1 - \alpha_H) \cdot \inf S$.*

**Proof.** It is easy to check that each bounded closed set $S$ contains its infimum $\underline{S} \stackrel{\text{def}}{=} \inf S$ and supremum $\underline{S} \stackrel{\text{def}}{=} \sup S$: $\{\underline{S}, \overline{S}\} \subseteq S \subseteq [\underline{S}, \overline{S}]$. Thus,

$$[2\underline{S}, 2\overline{S}] = \{\underline{S}, \overline{S}\} + [\underline{S}, \overline{S}] \subseteq S + [\underline{S}, \overline{S}] \subseteq [\underline{S}, \overline{S}] + [\underline{S}, \overline{S}] = [2\underline{S}, 2\overline{S}].$$

So, $S + [\underline{S}, \overline{S}] = [2\underline{S}, 2\overline{S}]$. By additivity, we conclude that $P(S) + P([\underline{S}, \overline{S}]) = P([2\underline{S}, 2\overline{S}])$. Due to conservativeness, we know the fair prices $P([\underline{S}, \overline{S}])$ and $P([2\underline{S}, 2\overline{S}])$. Thus, we can conclude that

$$P(S) = P([2\underline{S}, 2\overline{S}]) - P([\underline{S}, \overline{S}]) = (\alpha_H \cdot (2\overline{S}) + (1 - \alpha_H) \cdot (2\underline{S})) - (\alpha_H \cdot \overline{S} + (1 - \alpha_H) \cdot \underline{S}),$$

hence indeed $P(S) = \alpha_H \cdot \overline{S} + (1 - \alpha_H) \cdot \underline{S}$. The proposition is proven.

## 6 Case of Embedded Sets

In addition to a set $S$ of possible values of the gain $u$, we may also know a subset $s \subseteq S$ of more probable values $u$. To describe a fair price assigned to such a pair $(S, s)$, let us start with the simplest case when the original set $S$ is an interval $S = [\underline{u}, \overline{u}]$, and the subset $s$ is a single "most probable" value $u_0$ within this interval. Such pairs are known as *triples*; see, e.g., [1] and references therein. For triples, addition is defined component-wise:

$$([\underline{u}, \overline{u}], u_0) + ([\underline{v}, \overline{v}], v_0) = ([\underline{u} + \underline{v}, \overline{u} + \overline{v}], u_0 + v_0).$$

Thus, the additivity requirement about the fair prices takes the form

$$P([\underline{u} + \underline{v}, \overline{u} + \overline{v}], u_0 + v_0) = P([\underline{u}, \overline{u}], u_0) + P([\underline{v}, \overline{v}], v_0).$$

**Definition 3.** *By a* fair price under triple uncertainty*, we mean a function* $P([\underline{u}, \overline{u}], u_0)$ *for which:*

- $\underline{u} \le P([\underline{u}, \overline{u}], u_0) \le \overline{u}$ *for all* $\underline{u} \le u \le \overline{u}$ *(conservativeness);*
- *if* $\underline{u} \le \underline{v}$*,* $u_0 \le v_0$*, and* $\overline{u} \le \overline{v}$*, then* $P([\underline{u}, \overline{u}], u_0) \le P([\underline{v}, \overline{v}], v_0)$ *(monotonicity);*
- *(additivity) for all* $\underline{u}$*,* $\overline{u}$*,* $u_0$ $\underline{v}$*,* $\overline{v}$*, and* $v_0$*, we have*

$$P([\underline{u} + \underline{v}, \overline{u} + \overline{v}], u_0 + v_0) = P([\underline{u}, \overline{u}], u_0) + P([\underline{v}, \overline{v}], v_0).$$

**Proposition 3.** *Each fair price under triple uncertainty has the form*

$$P([\underline{u}, \overline{u}], u_0) = \alpha_L \cdot \underline{u} + (1 - \alpha_L - \alpha_U) \cdot u_0 + \alpha_U \cdot \overline{u}, \text{ where } \alpha_L, \alpha_U \in [0, 1].$$

**Proof.** In general, we have

$$([\underline{u}, \overline{u}], u_0) = ([u_0, u_0], u_0) + ([0, \overline{u} - u], 0) + ([\underline{u} - u, 0], 0).$$

So, due to additivity:

$$P([\underline{u}, \overline{u}], u_0) = P([u_0, u_0], u_0) + P([0, \overline{u} - u_0], 0) + P([\underline{u} - u_0, 0], 0).$$

Due to conservativeness, $P([u_0, u_0], u_0) = u_0$.

Similarly to the interval case, we can prove that $P([0, r], 0) = \alpha_U \cdot r$ for some $\alpha_U \in [0, 1]$, and that $P([r, 0], 0) = \alpha_L \cdot r$ for some $\alpha_L \in [0, 1]$. Thus,

$$P([\underline{u}, \overline{u}], u_0) = \alpha_L \cdot \underline{u} + (1 - \alpha_L - \alpha_U) \cdot u_0 + \alpha_U \cdot \overline{u}.$$

The proposition is proven.

The next simplest case is when both sets $S$ and $s \subseteq S$ are intervals, i.e., when, inside the interval $S = [\underline{u}, \overline{u}]$, instead of a "most probable" value $u_0$, we have a "most probable" subinterval $[\underline{m}, \overline{m}] \subseteq [\underline{u}, \overline{u}]$. The resulting pair of intervals is known as a "twin interval" (see, e.g., [4, 11]).

For such twin intervals, addition is defined component-wise:

$$([\underline{u}, \overline{u}], [\underline{m}, \overline{m}]) + ([\underline{v}, \overline{v}], [\underline{n}, \overline{n}]) = ([\underline{u} + \underline{v}, \overline{u} + \overline{v}], [\underline{m} + \underline{n}, \overline{m} + \overline{n}]).$$

Thus, the additivity requirement about the fair prices takes the form

$$P([\underline{u} + \underline{v}, \overline{u} + \overline{v}], [\underline{m} + \underline{n}, \overline{m} + \overline{n}]) = P([\underline{u}, \overline{u}], [\underline{m}, \overline{m}]) + P([\underline{v}, \overline{v}], [\underline{n}, \overline{n}]).$$

**Definition 4.** *By a* fair price under twin uncertainty*, we mean a function* $P([\underline{u}, \overline{u}], [\underline{m}, \overline{m}])$ *for which:*

- $\underline{u} \le P([\underline{u}, \overline{u}], [\underline{m}, \overline{m}]) \le \overline{u}$ *for all* $\underline{u} \le \underline{m} \le \overline{m} \le \overline{u}$ *(conservativeness);*
- *if* $\underline{u} \le \underline{v}$, $\underline{m} \le \underline{n}$, $\overline{m} \le \overline{n}$, *and* $\overline{u} \le \overline{v}$, *then* $P([\underline{u}, \overline{u}], [\underline{m}, \overline{m}]) \le P([\underline{v}, \overline{v}], [\underline{n}, \overline{n}])$ *(monotonicity);*
- *for all* $\underline{u} \le \underline{m} \le \overline{m} \le \overline{u}$ *and* $\underline{v} \le \underline{n} \le \overline{n} \le \overline{v}$, *we have additivity:*

$$P([\underline{u} + \underline{v}, \overline{u} + \overline{v}], [\underline{m} + \underline{n}, \overline{m} + \overline{m}]) = P([\underline{u}, \overline{u}], [\underline{m}, \overline{m}]) + P([\underline{v}, \overline{v}], [\underline{n}, \overline{n}]).$$

**Proposition 4.** *Each fair price under twin uncertainty has the following form, for some* $\alpha_L, \alpha_u, \alpha_U \in [0, 1]$*:*

$$P([\underline{u}, \overline{u}], [\underline{m}, \overline{m}]) = \underline{m} + \alpha_u \cdot (\overline{m} - \underline{m}) + \alpha_U \cdot (\overline{u} - \overline{m}) + \alpha_L \cdot (\underline{u} - \underline{m}).$$

**Proof.** In general, we have

$$([\underline{u}, \overline{u}], [\underline{m}, \overline{m}]) = ([\underline{m}, \underline{m}], [\underline{m}, \underline{m}]) + ([0, \overline{m} - \underline{m}], [0, \overline{m} - \underline{m}]) +$$

$$([0, \overline{u} - \overline{m}], [0, 0]) + ([\underline{u} - \underline{m}, 0], [0, 0)].$$

So, due to additivity:

$$P([\underline{u}, \overline{u}], [\underline{m}, \overline{m}]) = P([\underline{m}, \underline{m}], [\underline{m}, \underline{m}]) + P([0, \overline{m} - \underline{m}], [0, \overline{m} - \underline{m}]) +$$

$$P([0, \overline{u} - \overline{m}], [0, 0]) + P([\underline{u} - \underline{m}, 0], [0, 0)].$$

Due to conservativeness, $P([\underline{m}, \underline{m}], [\underline{m}, \underline{m}]) = \underline{m}$. Similarly to the interval case, we can prove that:

- $P([0, r], [0, r]) = \alpha_u \cdot r$ for some $\alpha_u \in [0, 1]$,
- $P([0, r], [0, 0]) = \alpha_U \cdot r$ for some $\alpha_U \in [0, 1]$;
- $P([r, 0], [0, 0]) = \alpha_L \cdot r$ for some $\alpha_L \in [0, 1]$.

Thus,

$$P([\underline{u},\overline{u}],[\underline{m},\overline{m}])=\underline{m}+\alpha_u\cdot(\overline{m}-\underline{m})+\alpha_U\cdot(\overline{u}-\overline{m})+\alpha_L\cdot(\underline{u}-\underline{m}).$$

The proposition is proven.

Finally, let us consider the general case.

**Definition 5.** *By a fair price under embedded-set uncertainty, we mean a function $P$ that assigns, to every pair of bounded closed sets $(S,s)$ with $s\subseteq S$, a real number $P(S,s)$, for which:*

- $P([\underline{u},\overline{u}],[\underline{m},\overline{m}])=\underline{m}+\alpha_u\cdot(\overline{m}-\underline{m})+\alpha_U\cdot(\overline{U}-\overline{m})+\alpha_L\cdot(\underline{u}-\underline{m})$ *(conservativeness);*
- $P(S+S',s+s')=P(S,s)+P(S',s')$ *(additivity).*

**Proposition 5.** *Each fair price under embedded-set uncertainty has the form*

$$P(S,s)=\inf s+\alpha_u\cdot(\sup s-\inf s)+\alpha_U\cdot(\sup S-\sup s)+\alpha_L\cdot(\inf S-\inf s).$$

**Proof.** Similarly to the proof of Proposition 2, we can conclude that

$$(S,s)+([\inf S,\sup S],[\inf s,\sup s])=([2\cdot\inf S,2\cdot\sup S],[2\cdot\inf s,2\cdot\sup s]).$$

By additivity, we conclude that

$$P(S,s)+P([\inf S,\sup S],[\inf s,\sup s])=$$

$$P([2\cdot\inf S,2\cdot\sup S],[2\cdot\inf s,2\cdot\sup s]),$$

hence

$$P(S,s)=P([2\cdot\inf S,\cdot\sup S],[2\cdot\inf s,2\cdot\sup s])-$$

$$P([\inf S,\sup S],[\inf s,\sup s]).$$

Due to conservativeness, we know the fair prices

$$P([2\cdot\inf S,2\cdot\sup S],[2\cdot\inf s,2\cdot\sup s])\text{ and }P([\inf S,\sup S],[\inf s,\sup s]).$$

Subtracting these expressions, we get the desired formula for $P(S,s)$. The proposition is proven.

## 7 Cases of Probabilistic and p-Box Uncertainty

Suppose that for some financial instrument, we know the corresponding probability distribution $F(u)$ on the set of possible gains $u$. What is the fair price $P$ for this instrument?

Due to additivity, the fair price for $n$ copies of this instrument is $n \cdot P$. According to the Large Numbers Theorem, for large $n$, the average gain tends to the mean value $\mu = \int u\, dF(u)$.

Thus, the fair price for $n$ copies of the instrument is close to $n \cdot \mu$: $n \cdot P \approx n \cdot \mu$. The larger $n$, the closer the averages. So, in the limit, we get $P = \mu$.

So, the fair price under probabilistic uncertainty is equal to the average gain $\mu = \int u\, dF(u)$.

Let us now consider the case of a p-box $[\underline{F}(u), \overline{F}(u)]$. For different functions $F(u) \in [\underline{F}(u), \overline{F}(u)]$, values of the mean $\mu$ form an interval $\left[\underline{\mu}, \overline{\mu}\right]$, where $\underline{\mu} = \int u\, d\overline{F}(u)$ and $\overline{\mu} = \int u\, d\underline{F}(u)$. Thus, the price of a p-box is equal to the price of an interval $\left[\underline{\mu}, \overline{\mu}\right]$.

We already know that the fair price of this interval is equal to

$$\alpha_H \cdot \overline{\mu} + (1 - \alpha_H) \cdot \underline{\mu}.$$

Thus, we conclude that the fair price of a p-box $[\underline{F}(u), \overline{F}(u)]$ is $\alpha_H \cdot \overline{\mu} + (1 - \alpha_H) \cdot \underline{\mu}$, where $\underline{\mu} = \int u\, d\overline{F}(u)$ and $\overline{\mu} = \int u\, d\underline{F}(u)$.

## 8 Case of Kaucher (Improper) Intervals

For Kaucher intervals, addition is also defined component-wise; in particular, for all $\underline{u} < \overline{u}$, we have

$$[\underline{u}, \overline{u}] + [\overline{u}, \underline{u}] = [\underline{u} + \overline{u}, \underline{u} + \overline{u}].$$

Thus, additivity implies that

$$P([\underline{u}, \overline{u}]) + P([\overline{u}, \underline{u}]) = P([\underline{u} + \overline{u}, \underline{u} + \overline{u}]).$$

We know that $P([\overline{u}, \underline{u}]) = \alpha_H \cdot \underline{u} + (1 - \alpha_H) \cdot \overline{u}$ and $P([\underline{u} + \overline{u}, \underline{u} + \overline{u}]) = \underline{u} + \overline{u}$. Hence:

$$P([\underline{u}, \overline{u}]) = (\underline{u} + \overline{u}) - (\alpha_H \cdot \underline{u} + (1 - \alpha_H) \cdot \overline{u}).$$

Thus, the fair price $P([\underline{u}, \overline{u}])$ of an improper interval $[\underline{u}, \overline{u}]$, with $\underline{u} > \overline{u}$, is equal to $P([\underline{u}, \overline{u}]) = \alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u}$.

## 9 Summary and Conclusions

In this paper, for different types of uncertainty, we derive the formulas for the fair prices under reasonable conditions of conservativeness, monotonicity, and additivity.

In the simplest case of interval uncertainty, when we only know the interval $[\underline{u}, \overline{u}]$ of possible values of the gain $u$, the fair price is equal to

$$P([\underline{u}, \overline{u}]) = \alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u},$$

for some parameter $\alpha_H \in [0, 1]$. Thus, the fair price approach provides a justification for the formula originally proposed by a Nobelist L. Hurwicz, in which $\alpha_H$ describes the decision maker's optimism degree: $\alpha_H = 1$ corresponds to pure optimism, $\alpha_H = 0$ to pure pessimism, and intermediate values of $\alpha_H$ correspond to a realistic approach that takes into account both best-case (optimistic) and worst-case (pessimistic) scenarios.

In a more general situation, when the set $S$ of possible values of the gain $u$ is not necessarily an interval, the fair price is equal to

$$P(S) = \alpha_H \cdot \sup S + (1 - \alpha_H) \cdot \inf(S).$$

If, in addition to the set $S$ of possible values of the gain $u$, we also know a subset $s \subseteq S$ of "most probable" gain values, then the fair price takes the form

$$P(S, s) = \inf s + \alpha_u \cdot (\sup s - \inf s) + \alpha_U \cdot (\sup S - \sup s) + \alpha_L \cdot (\inf S - \inf s),$$

for some values $\alpha_u$, $\alpha_L$, and $\alpha_U$ from the interval $[0, 1]$. In particular, when both sets $S$ and $s$ are intervals, i.e., when $S = [\underline{u}, \overline{u}]$ and $s = [\underline{m}, \overline{m}]$, the fair price takes the form

$$P([\underline{u}, \overline{u}], [\underline{m}, \overline{m}]) = \underline{m} + \alpha_u \cdot (\overline{m} - \underline{m}) + \alpha_U \cdot (\overline{u} - \overline{m}) + \alpha_L \cdot (\underline{u} - \underline{m}).$$

When the interval $s$ consists of a single value $u_0$, this formula turns into

$$P([\underline{u}, \overline{u}], u_0) = \alpha_L \cdot \underline{u} + (1 - \alpha_L - \alpha_U) \cdot u_0 + \alpha_U \cdot \overline{u}.$$

When, in addition to the set $S$, we also know the cumulative distributive function (cdf) $F(u)$ that describes the probability distribution of different possible values $u$, then the fair price is equal to the expected value of the gain

$$P(F) = \int u \, dF(u).$$

In situations when for each $u$, we only know the interval $[\underline{F}(u), \overline{F}(u)]$ of possible values of the cdf $F(u)$, then the fair price is equal to

$$P([\underline{F}, \overline{F}]) = \alpha_H \cdot \int u \, d\overline{F}(u) + (1 - \alpha_H) \cdot \int u \, d\underline{F}(u).$$

Finally, when uncertainty is described by an improper interval $[\underline{u}, \overline{u}]$ with $\underline{u} > \overline{u}$, the fair price is equal to

$$P([\underline{u}, \overline{u}]) = \alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u}.$$

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## References

1. Cole, A.J., Morrison, R.: Triplex: a system for interval arithmetic, Software: Practice and Experience 12(4), 341–350 (1982).
2. Ferson, S.: Risk Assessment with Uncertainty Numbers: RiskCalc, CRC Press, Boca Raton, Florida (2002)
3. Ferson, S., Kreinovich, V., Oberkampf, W., Ginzburg, L.: Experimental Uncertainty Estimation and Statistics for Data Having Interval Uncertainty, Sandia National Laboratories, Report SAND2007-0939 (2007)
4. Gardefies, E., Trepat, A., Janer, J.M.: SIGLA-PL/1: development and applications, In: Nickel, K. L. E. (ed.), Interval Mathematics 1980, Academic Press, New York, 301–315 (1980)
5. Hurwicz, L.: Optimality criteria for decision making under ignorance, Cowles Commission Discussion Paper, Statistics, No. 370 (1951)
6. Jaulin, L., Kieffer, M., Didrit, O., Walter, E.: Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control and Robotics, Springer-Verlag, London (2001)
7. Kaucher, E.: Über Eigenschaften und Anwendungsmöglichkeiten der erweiterten lntervallrechnung und des hyperbolische Fastköpers über $R$, Computing, Supplement 1, 81–94 (1977)
8. Luce, R. D., Raiffa, R.: Games and Decisions: Introduction and Critical Survey, Dover, New York (1989)
9. McKee, J., Lorkowski, J., Ngamsantivong, T.: Note on fair price under interval uncertainty, Journal of Uncertain Systems 8(3), 186–189 (2014)
10. Moore, R.E., Kearfott, R.B., Cloud, M.J. Introduction to Interval Analysis, SIAM Press, Philadelphia, Pennsylviania (2009)
11. Nesterov, V.M.: Interval and twin arithmetics, Reliable Computing 3(4), 369–380 (1997)
12. Raiffa, H.: Decision Analysis: Introductory Lectures on Choices Under Uncertainty, Mcgraw-Hill (1997)
13. Sainz, M.A., Armengol, J., Calm, R., Herrero, P., Jorba, L., Vehi, J.: Modal Interval Analysis, Springer, Berlin, Heidelberg, New York (2014)