Sometimes, It Is Beneficial to Process Different Types of Uncertainty Separately

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***Abstract*—In many practical situations, we make predictions based on the measured and/or estimated values of different physical quantities. The accuracy of these predictions depends on the accuracy of the corresponding measurements and expert estimates. Often, for each quantity, there are several different sources of inaccuracy. Usually, to estimate the prediction accuracy, we first combine, for each input, inaccuracies from different sources into a single expression, and then use these expressions to estimate the prediction accuracy. In this paper, we show that it is often more computationally efficient to process different types of uncertainty separately, i.e., to estimate inaccuracies in the prediction result caused by different types of uncertainty, and only then combine these inaccuracies into a single estimate.**

I. Estimating Accuracy of Data Processing: A General Reminder

What we plan to do: structure of this paper. In this paper, we show that we can often speed up the estimation of accuracy of data processing. To explain our idea, we first describe, in this section, the general ideas and techniques of estimating accuracy of data processing. In Section 2, we then describe the case for which our idea is intended: a frequent case when we have inaccuracies of different types. Finally, in Section 3, we explain our idea and show that the use of this idea can indeed speed up computations.

Need for data processing. One of the main objectives of science is to predict future values of physical quantities. For example:

- in meteorology, we need to predict future weather;
- in airplane control, we need to predict the location and the velocity of the plane under current control, etc.

To make this prediction of a future value y, we need to know the relation $y = f(x_1, \ldots, x_n)$ between the desired future value y and the current values of related physical quantities $x_1, \ldots, x_n$. In these terms, the prediction consists of two stages:

- first, we measure or estimate the values of the quantities $x_1, \ldots, x_n$;
- then, we use the results $\widetilde{x}_i$ of measurement or estimation to compute an estimate $\widetilde{y}$ of the desired future value y as

$$\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n).$$

This computation is an important case of *data processing*.

Need to take uncertainty into account. Measurements are never absolutely accurate, and expert estimates are even less accurate. As a result, the estimates $\widetilde{x}_i$ are, in general, different from the actual (unknown) values x_i of the corresponding quantities. Therefore, the estimate $\widetilde{y}$ is also only approximate.

In practice, it is desirable to know how accurate is this estimate $\widetilde{y}$. To find this out, we need to take into account the accuracy of the estimates $\widetilde{x}_i$.

How to gauge the accuracy of the estimates $\widetilde{x}_i$. In this paper, we consider two types of estimates: measurements and expert estimates.

For measurements, we usually know the upper bound Δ_i on the absolute value of the measurement error $\Delta x_i \stackrel{\text{def}}{=} \widetilde{x}_i - x_i$; see, e.g., [7]. This upper bound is usually provided by the manufacturer of the measurement instrument. The existence of such an upper bound comes from the very nature of measurement: if no upper bound is guaranteed, this means that whatever result the "measuring instrument" produces, the actual value can be any number from $-\infty$ to $+\infty$; this would be not a measurement result, it would be a wild guess.

Once we know the upper bound Δ_i for which $|\Delta x_i| \le \Delta_i$, and we know the measurement result $\widetilde{x}_i$, then we know that the actual value x_i is located in the interval $[\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$.

To gauge the accuracy of fuzzy estimates, it is reasonable to use fuzzy techniques, techniques specifically designed to describe imprecise ("fuzzy") expert estimates in precise computer-understandable terms; see, e.g., [1], [6], [9]. In these techniques, the uncertainty of each estimate is described by a *membership function* $\mu_i(x_i)$ that describes, for all possible real numbers x_i, the degree to which the expert believes that this number is a possible value of the corresponding quantity.

Comment. In some situations, in addition to the upper bound Δ_i on the measurement error, we also know the probabilities of different values $\Delta x_i \in [-\Delta_i, \Delta_i]$; see, e.g., [2], [3], [7]. We plan to analyze this case in the future.

How accurate is the corresponding model. In the ideal case, when the model used for prediction is exact, the actual future value y of the predicted quantity is equal to $y = f(x_1, \ldots, x_n)$.

In many practical situations, the model is approximate, i.e., there is, in general, a non-zero model inaccuracy

$$\Delta m \stackrel{\text{def}}{=} y - f(x_1, \ldots, x_n).$$

In this case, $y = f(x_1, \ldots, x_n) + \Delta m$.

What do we know about the model inaccuracy Δm? Similarly to the inaccuracy of estimations, there are two possible situations:

- in some cases, we know the upper bound Δ_m on the absolute value of the model inaccuracy Δm:

$$|\Delta m| \le \Delta_m;$$

- in other cases, we know a membership function $\mu_m(\Delta m)$ that describes, for each real number Δm, the degree of possibility that the model inaccuracy is equal to this number.

Measurement and estimation inaccuracies are usually small. In many practical situations, the measurement and estimation inaccuracies Δx_i are relatively small, so that we can safely ignore terms which are quadratic (or of higher order) in terms of Δx_i [7]. We can use this fact to simplify the expression for the inaccuracy $\Delta y \stackrel{\text{def}}{=} \widetilde{y} - y$.

Here, by definition of data processing, $\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$, and by definition of model inaccuracy,

$$y = f(x_1, \ldots, x_n) + \Delta m.$$

Thus,

$$\Delta y = \widetilde{y} - y = f(\widetilde{x}_1, \ldots, \widetilde{x}_n) - f(x_1, \ldots, x_n) - \Delta m. \quad (1)$$

From the definition of the measurement uncertainty Δx_i, we conclude that $x_i = \widetilde{x}_i - \Delta x_i$. Substituting this expression into the above formula (1) for Δy, we conclude that

$$\Delta y = \widetilde{y} - y =$$

$$f(\widetilde{x}_1, \ldots, \widetilde{x}_n) - f(\widetilde{x}_1 - \Delta x_1, \ldots, \widetilde{x}_n - \Delta x_n) - \Delta m. \quad (2)$$

Expanding the expression $f(\widetilde{x}_1 - \Delta x_1, \ldots, \widetilde{x}_n - \Delta x_n)$ in Taylor series in terms of small values Δx_i, and using the fact that terms quadratic in Δx_i can be safely ignored, we conclude that

$$f(\widetilde{x}_1 - \Delta x_1, \ldots, \widetilde{x}_n - \Delta x_n) =$$

$$f(\widetilde{x}_1, \ldots, \widetilde{x}_n) - \sum_{i=1}^{n} c_i \cdot \Delta x_i, \quad (3)$$

where we denoted

$$c_i \stackrel{\text{def}}{=} \frac{\partial f}{\partial x_i}. \quad (4)$$

Substituting the expression (3) into the formula (2) and cancelling out the terms $+f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ and $-f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$, we conclude that

$$\Delta y = \sum_{i=1}^{n} c_i \cdot \Delta x_i - \Delta m. \quad (5)$$

This is the main formula used to estimate the accuracy of estimating the desired quantity y.

How to estimate partial derivatives. The values of the partial derivatives c_i can be determined either by an explicit differentiation – if we have an explicit expression for $f(x_1, \ldots, x_n)$, or by numerical differentiation

$$c_i \approx \frac{f(\widetilde{x}_1, \ldots, \widetilde{x}_{i-1}, \widetilde{x}_i + h, \widetilde{x}_{i+1}, \ldots, \widetilde{x}_n) - \widetilde{y}}{h} \quad (6)$$

for some small h.

Comment. The formula for numerical differentiation is easy to explain, it naturally comes from the definition of the partial derivative as the limit of the corresponding ratios:

$$c_i = \frac{\partial f}{\partial x_i} \stackrel{\text{def}}{=}$$

$$\lim_{h \to 0} \frac{f(\widetilde{x}_1, \ldots, \widetilde{x}_{i-1}, \widetilde{x}_i + h, \widetilde{x}_{i+1}, \ldots, \widetilde{x}_n) - \widetilde{y}}{h}. \quad (7)$$

By the definition of the limit, this means that when h is small, the right-hand side of the formula (7) is close to the partial derivative, and the smaller h, the closer is this ratio to c_i. Thus, if we take a sufficiently small value h, we can safely use this ratio as an accurate estimate for the partial derivative c_i.

How to estimate Δy: case of interval uncertainty. Let us assume that we have interval bounds for all the estimation errors Δx_i and Δm. In this case:

- each estimation error Δx_i can take any value between $-\Delta_i$ and Δ_i, and
- the model inaccuracy Δm can take any value between $-\Delta_m$ and Δ_m.

The largest possible value of Δy is attained when each of the terms $c_i \cdot \Delta x_i$ and $-\Delta m$ in the sum (5) attains its largest possible value.

- When $c_i > 0$, the function $c_i \cdot \Delta x_i$ is increasing and thus, it largest value is attained when Δx_i attains its largest value Δ_i. The resulting value of $c_i \cdot \Delta x_i$ is equal to $c_i \cdot \Delta_i$.
- When $c_i < 0$, the function $c_i \cdot \Delta x_i$ is decreasing and thus, it largest value is attained when Δx_i attains its smallest value $-\Delta_i$. The resulting value of $c_i \cdot \Delta x_i$ is equal to $c_i \cdot (-\Delta_i) = (-c_i) \cdot \Delta_i$.

In both cases, the largest value of the term $c_i \cdot \Delta x_i$ is equal to $|c_i| \cdot \Delta_i$.

Similarly, the largest value of $-\Delta m$ is attained when Δm attains its smallest possible value $-\Delta_m$. The corresponding value of $-\Delta m$ is thus equal to $-(-\Delta_m) = \Delta_m$. Thus, the largest value Δ of the sum (5) is equal to

$$\Delta = \sum_{i=1}^{n} |c_i| \cdot \Delta_i + \Delta_m. \quad (8)$$

Similarly, the smallest possible value of Δy is attained when each of the terms $c_i \cdot \Delta x_i$ and $-\Delta m$ in the sum (5) attains its smallest possible value.

- When $c_i > 0$, the function $c_i \cdot \Delta x_i$ is increasing and thus, it smallest value is attained when Δx_i attains its

smallest value $-\Delta_i$. The resulting value of $c_i \cdot \Delta x_i$ is equal to $-c_i \cdot \Delta_i$.

- When $c_i < 0$, the function $c_i \cdot \Delta x_i$ is decreasing and thus, it smallest value is attained when Δx_i attains its largest value Δ_i. The resulting value of $c_i \cdot \Delta x_i$ is equal to $c_i \cdot \Delta_i$.

In both cases, the smallest value of the term $c_i \cdot \Delta x_i$ is equal to $-|c_i| \cdot \Delta_i$.

Similarly, the smallest value of $-\Delta m$ is attained when Δm attains its largest possible value Δ_m. The corresponding value of $-\Delta m$ is thus equal to $-\Delta_m$. Thus, the smallest value Δ of the sum (5) is equal to

$$-\sum_{i=1}^{n} |c_i| \cdot \Delta - \Delta_m. \tag{9}$$

One can see that this is exactly $-\Delta$, where Δ is described by the formula (8).

How to estimate Δy: case of interval uncertainty – resulting formula. In case of interval uncertainty, possible values of Δy form an interval $[-\Delta, \Delta]$, where Δ is described by the formula (8).

Comment. It should be mentioned that when the number n of inputs is large, then instead of directly using the formula (8), we can use a faster Monte-Carlo type algorithm based on Cauchy distributions; see, e.g., [4].

How to estimate Δy: case of fuzzy uncertainty. Let us assume that we have fuzzy estimates $\mu_i(\Delta x_i)$ and $\mu_m(\Delta m)$ for all the estimation errors Δx_i and Δm. In this case, we want to estimate, for every real number Δy, the degree $\mu(\Delta y)$ to which this number is a possible value of data processing inaccuracy.

The value Δy is a possible value of inaccuracy if there exist values Δx_i and Δm

- which are possible as inaccuracies of input estimation and model, and
- for which the formula (5) holds.

For simplicity, let us use $\min(a, b)$ to describe "and", and $\max(a, b)$ to describe "or". Then, for each combination of values Δx_i and Δm, the degree to which all these values are possible is equal to the minimum of the degrees to which each of them is possible:

$$\min(\mu_1(\Delta x_1), \ldots \mu_n(\Delta x_n), \mu_m(\Delta m)),$$

and the desired degree Δy is equal to the maximum of these expressions over all possible combinations of Δx_i and Δ_m:

$$\mu(\Delta y) =$$

$$\max \min(\mu_1(\Delta x - 1), \ldots \mu_n(\Delta x_n), \mu_m(\Delta m)), \tag{10}$$

where the maximum is taken over all tuples that satisfy the formula (5); this expression is known as *Zadeh's extension principle*.

It is know that the computation of this membership function can be simplified if instead of each membership function $\mu(z)$, we consider its *α-cuts*, i.e., sets ${}^{\alpha}z \stackrel{\text{def}}{=} \{z : \mu(z) \geq \alpha\}$ corresponding to different values $\alpha \in [0, 1]$.

It should be mentioned that since $\mu(z)$ is always non-negative, the above definition (with non-strict inequality $\mu(z) \geq \alpha$) does not work for $\alpha = 0$: strictly speaking, all real numbers z satisfy the corresponding inequality. Thus, for $\alpha = 0$, the α-cut is defined slightly differently: as the closure of the set $\{z : \mu(z) > 0\}$ corresponding to strict inequality.

Usually, membership functions correspond to *fuzzy numbers*, i.e., all α-cuts are intervals. Moreover, the α-cuts corresponding to Δx_i and Δ_m are usually symmetric, i.e., have the form ${}^{\alpha}\Delta x_i = [-{}^{\alpha}\Delta_i, {}^{\alpha}\Delta_i]$ and ${}^{\alpha}\Delta m = [-{}^{\alpha}\Delta_m, {}^{\alpha}\Delta_m]$ for appropriate values ${}^{\alpha}\Delta_i$ and ${}^{\alpha}\Delta_m$.

One can easily check, based on the formula (10), that $\mu(\Delta) \geq \alpha$ if and only if there exists a tuple $(\Delta x_1, \ldots, \Delta x_n, \Delta_m)$ for which

$$\min(\mu_1(\Delta x_1), \ldots \mu_n(\Delta x_n), \mu_m(\Delta m)) \geq \alpha,$$

i.e., equivalently, for which $\mu_i(\Delta_i) \geq \alpha$ for all i and $\mu_m(\Delta_m) \geq \alpha$.

In other words, Δy belongs to the α-cut if and only if it is a possible value of the expression (5) when:

- Δx_i belongs to the corresponding α-cut $[-{}^{\alpha}\Delta_i, {}^{\alpha}\Delta_i]$ and
- Δm belongs to the α-cut $[-{}^{\alpha}\Delta_m, {}^{\alpha}\Delta_m]$.

We already know how to compute the range of such values Δy. Thus, we arrive at the following algorithm for computing the desired α-cut $[-{}^{\alpha}\Delta, {}^{\alpha}\Delta]$.

How to estimate Δy: case of fuzzy uncertainty – resulting formula. In case of fuzzy uncertainty, for every $\alpha \in [0, 1]$, we are given the α-cuts $[-{}^{\alpha}\Delta_i, {}^{\alpha}\Delta_i]$ and $[-{}^{\alpha}\Delta_m, {}^{\alpha}\Delta_m]$ describing the expert's uncertainty about the estimates $\widetilde{x}_i$ and about the model used in data processing.

Based on these α-cuts, we can compute the α-cuts $[-{}^{\alpha}\Delta, {}^{\alpha}\Delta]$ for Δy as follows:

$${}^{\alpha}\Delta = \sum_{i=1}^{n} |c_i| \cdot {}^{\alpha}\Delta_i + {}^{\alpha}\Delta_m. \tag{11}$$

Comment. In principle, there are infinitely many different values α in the interval $[0, 1]$. However, we should take into account that the values α correspond to experts' degrees of confidence, and experts cannot describe their degrees with too much accuracy.

Usually, it is sufficient to consider only eleven values $\alpha = 0.0$, $\alpha = 0.1$, $\alpha = 0.2$, …, $\alpha = 0.9$, and $\alpha = 1.0$. Thus, we need to apply the formula (11) eleven times.

This is in line with the fact that, as psychologists have found, we usually divide each quantity into 7 plus plus minus 2 categories – this is the largest number of categories whose meaning we can immediately grasp; see, e.g., [5], [8]. For

some people, this "magical number" is $7+2=9$, for some it is $7-2=5$. So, to make sure that do not miss on some people's subtle divisions, it is sufficient to have at least 9 different categories. From this viewpoint, 11 categories are sufficient; usually the above eleven values are chosen since for us, it is easier to understand decimal numbers.

II. Case for Which Simplification Is Possible: A Description

In this section, we describe the cases for which the computations can be simplified.

Simplest case: when all fuzzy numbers are of the same type. sometimes, all membership functions are "of the same type", i.e., they all have the form $\mu(z)=\mu_0(k\cdot z)$ for some fixed symmetric function $\mu_0(z)$.

For example, frequently, we consider symmetric triangular functions; all these functions can be obtained from the standard triangular function $\mu_0(z)=\max(1-|z|,0)$ by using an appropriate constant $k>0$.

In this case, we can simplify computations. Let us show that in this simple case, we can drastically reduce the computation time that is needed to compute the desired α-cuts ${}^{\alpha}\Delta$.

Indeed, let $[-{}^{\alpha}\Delta_0, {}^{\alpha}\Delta_0]$ denote an α-ut corresponding to the membership function $\mu_0(z)$. This means that the inequality $\mu_0(z)\ge\alpha$ is equivalent to $|z|\le {}^{\alpha}\Delta_0$. Then, for the membership function $\mu(z)=\mu_0(k\cdot z)$, the inequality $\mu(z)\ge\alpha$ describing its α-cut is equivalent to $\mu_0(k\cdot z)\ge\alpha$, i.e., to to $k\cdot|z|\le {}^{\alpha}\Delta_0$ and thus, $|z|\le\dfrac{1}{k}\cdot{}^{\alpha}\Delta_0$. Hence, the half-widths of the corresponding α-cuts are equal to

$$ {}^{\alpha}\Delta=\frac{1}{k}\cdot{}^{\alpha}\Delta_0. \tag{12}$$

This equality holds for all α, in particular, for $\alpha=0$, when we get

$$ {}^{0}\Delta=\frac{1}{k}\cdot{}^{0}\Delta_0. \tag{13}$$

By dividing (12) by (13), we conclude that

$$\frac{{}^{\alpha}\Delta}{{}^{0}\Delta}=f(\alpha),$$

where we denoted

$$f(\alpha)\stackrel{\text{def}}{=}\frac{{}^{\alpha}\Delta_0}{{}^{0}\Delta_0}.$$

For example, for a triangular membership function, we have

$$f(\alpha)=1-\alpha.$$

Thus, if we know the type of the membership function (and hence, the corresponding function $f(\alpha)$), and we know the 0-cut, then we can reconstruct all α-cuts as

$$ {}^{\alpha}\Delta=f(\alpha)\cdot{}^{0}\Delta, \tag{14}$$

i.e., by simply multiplying the 0-cuts by an appropriate factor $f(\alpha)$.

So, if all the membership functions $\mu_i(\Delta x_i)$ and $\mu_m(\Delta m)$ are of the same type, then, for every α, we have ${}^{\alpha}\Delta_i=f(\alpha)\cdot{}^{0}\Delta_i$ and

$${}^{\alpha}\Delta_m=f(\alpha)\cdot{}^{0}\Delta_m.$$

Substituting these expressions into the formula (11), we conclude that

$${}^{\alpha}\Delta=\sum_{i=1}^{n}|c_i|\cdot f(\alpha)\cdot{}^{0}\Delta_i+f(\alpha)\cdot{}^{0}\Delta_m=$$

$$f(\alpha)\cdot\left(\sum_{i=1}^{n}|c_i|\cdot{}^{0}\Delta_i+{}^{0}\Delta_m\right),$$

i.e., that

$$ {}^{\alpha}\Delta=f(\alpha)\cdot{}^{0}\Delta. \tag{15}$$

Thus, if all the membership functions are of the same type $\mu_0(z)$, there is no need to apply the formula (11) eleven times: it is sufficient to compute it only once, e.g., for $\alpha=0$. To find all other values ${}^{\alpha}\Delta$, we can then simply multiply the resulting value ${}^{0}\Delta$ by the factors $f(\alpha)$ corresponding to the type $\mu_0(z)$.

A more general case. A more general case is when we have a list of T different types of uncertainty – i.e., types of membership functions – and each approximation error Δx_i consists of $\le T$ components of the corresponding type. In other words, for each i, we have

$$\Delta x_i=\sum_{t=1}^{T}\Delta x_{i,t} \tag{16}$$

and

$$\Delta m=\sum_{t=1}^{T}\Delta m_t, \tag{17}$$

where $\Delta x_{i,t}$ and Δm_t are uncertainties of the t-th type, and we know the corresponding membership functions $\mu_{i,t}(\Delta x_{i,t})$ and $\mu_{m,t}(\Delta m_t)$.

For example, type $t=1$ may correspond to intervals (which are, of course, a particular case of fuzzy uncertainty), type $t=2$ may correspond to triangular membership functions, etc.

How this case is processed now.

- First, we use the known membership functions $\mu_{i,t}(\Delta x_{i,t})$ and $\mu_{m,t}(\Delta m_t)$ to find the memberships functions $\mu_i(\Delta x_i)$ and $\mu_m(\Delta m)$ that correspond to the sums (16) and (17).
- Then, we use these membership functions to compute the desired membership function $\mu(\Delta y)$.

On the second stage, we follow the above algorithm, i.e., we apply the formula (11) eleven times.

III. Our Main Idea

Main idea. As we have mentioned, at present, to find the membership function for Δy, we use the formula (5), in which each of the terms Δx_i and Δm is computed by using the formulas (16) and (17).

A natural alternative idea is:

- to substitute the expressions (16) and (17) into the formula (5), and then

- to regroup the resulting terms by combining all the components of the same type t.

Technical details. Substituting the expressions (16) and (17) into the formula (5), we conclude that

$$\Delta y = \sum_{i=1}^{n} c_i \cdot \left(\sum_{t=1}^{T} \Delta x_{i,t} \right) - \left(\sum_{t=1}^{T} \Delta m_t \right). \qquad (18)$$

Now, grouping together all the terms corresponding to each type t, we conclude that

$$\Delta y = \sum_{t=1}^{T} \Delta y_t, \qquad (19)$$

where

$$\Delta y_t \stackrel{\text{def}}{=} \sum_{i=1}^{n} c_i \cdot \Delta x_{i,t} - \Delta m_t. \qquad (20)$$

This representation suggests the following new algorithm.

New algorithm: idea. For each t, since we are combining membership functions of the same type, computing these membership functions requires a single application of the formula (11), to compute the value ${}^0\Delta_t$ corresponding to $\alpha = 0$. The values corresponding to other values α, we simply multiply this value ${}^0\Delta_t$ by the coefficients $f_t(\alpha)$ corresponding to membership functions of type t.

Then, we add the resulting membership functions – by adding the corresponding α-cuts. Let us describe the resulting algorithm in detail.

New algorithm: in detail. We start with the values ${}^0\Delta_{i,t}$ and ${}^0\Delta_{m,t}$ for which the corresponding symmetric intervals $[-{}^0\Delta_{i,t}, {}^0\Delta_{i,t}]$ and $[-{}^0\Delta_{m,t}, {}^0\Delta_{m,t}]$ describe the 0-cuts of the corresponding membership functions $\mu_{i,t}(\Delta x_{i,t})$ and $\mu_{m,t}(\Delta m_t)$.

Based on these 0-cuts, we compute, for each type t, the values

$${}^0\Delta = \sum_{i=1}^{n} |c_i| \cdot {}^0\Delta_{i,t} + {}^0\Delta_{m,t}. \qquad (21)$$

Then, for $\alpha = 0$, $\alpha = 0.1$, …, and for $\alpha = 1.0$, we compute the values

$${}^\alpha\Delta_t = f_t(\alpha) \cdot {}^0\Delta_t, \qquad (22)$$

where the function $f_t(\alpha)$ is known for each type t. Finally, we add up α-cuts corresponding to different types t, to come up with the expression

$${}^\alpha\Delta = \sum_{t=1}^{T} {}^\alpha\Delta_t. \qquad (23)$$

Comment. We can combine the steps (22) and (23) into a single step, in which we use the following formula:

$${}^\alpha\Delta = \sum_{t=1}^{T} f_t(\alpha) \cdot {}^0\Delta_t. \qquad (24)$$

This new algorithm is much faster. The original algorithm computed the formula (11) eleven times. The new algorithm uses the corresponding formula (21) (the analogue of the formula (11)) T times, i.e., as many times as there are types. All the other computations are much faster, since they do not grow with the input size n.

Thus, if the number T of different types is smaller than eleven, the new methods is much faster.

For example, if we have $T = 2$ different types, e.g., intervals and triangular membership functions, then we get a $\dfrac{11}{2} = 5.5$ times speedup.

Conclusion. We can therefore conclude that sometimes, it is beneficial to process different types of uncertainty separately – namely, it is beneficial when we have ten or fewer different types of uncertainty. The fewer types of uncertainty we have, the faster the resulting algorithm.

Future work. In this paper, we show the handling uncertainty by types of uncertainty can make computations faster. We plan to test this idea of several actual data processing examples, to measure the actual gain, and thus, to make proper recommendations on when this idea is indeed beneficial. We also plan to extend this idea to other types of uncertainty, in particular, to different types of probabilistic uncertainty.

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