A Simplified Explanation of What It Means to Assign a Finite Value to an Infinite Sum

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***Abstract*—Recently, a video made rounds that explained that it often makes sense to assign finite values to infinite sums. For example, it makes sense to claim that the sum of all natural numbers is equal to $-1/12$. This has picked up interested in media. However, judged by the viewers' and readers' comments, for many viewers and readers, neither the video, not the corresponding articles seem to explain the meaning of the above inequality clearly enough. One of the main stumbling blocks is the fact that the infinite sum is clearly divergent, so a natural value of the infinite sum is infinity. What is the meaning of assigning a finite value to this (clearly infinite) sum? While the explanation of the above equality is difficult to describe in simple terms, the main idea behind this equality can be, in our opinion, explained rather naturally, and this is what we do in this paper.**

I. Introduction

The sum of all positive integers is equal to $-1/12$: a statement. A recent video [8] explains that it often makes sense to assign finite values to infinite sums. For example, it makes sense to claim that

$$1 + 2 + 3 + 4 + \ldots = -\frac{1}{12}.$$

This has picked up interested in other media sources, including an article in the New York Times [7].

A clarification is needed. Judged by the viewers' and readers' comments, for many viewers and readers, neither the video, not the article seem to explain the meaning of the above inequality clearly enough. One of the main stumbling blocks is the fact that the infinite sum

$$1 + 2 + \ldots$$

is clearly divergent: as n increases, the sum

$$1 + 2 + \ldots + n = \frac{n \cdot (n+1)}{2}$$

tends to infinity, so a natural value of the infinite sum is infinity.

What is the meaning of assigning a finite value to this (clearly infinite) sum?

What we do in this paper. While the explanation of the above equality is difficult to describe in simple terms, the main idea behind this equality can be, in our opinion, explained rather naturally. This is what we do in this paper.

II. How Can a Divergent Infinite Series Have a Finite Sum: A Simple Example

An example. To explain how a divergent infinite series can be assigned a finite sum, let us start with trying to define the sum of another infinite series:

$$1 + 2^1 + 2^2 + 2^3 + 2^4 + \ldots =$$

$$1 + 2 + 4 + 8 + 16 + \ldots$$

If we simply keep adding these numbers, we get larger and larger values:

- we start with 1,
- then, we get $1 + 2 = 3$,
- after that, we get $(1 + 2) + 4 = 3 + 4 = 7$,
- etc.

In the limit $n \to \infty$, the sum $1 + 2^2 + \ldots + 2^n$ tends to infinity.

Question. In what sense can we then declare some finite number to be the sum of the above infinite series?

Main idea. To be able to explain how the sum can be finite, let us do the following:

- instead of considering just the infinite series itself,
- let us consider a *family* of infinite series – of which includes the desired series is a particular case.

Let us apply this general idea to our simple example. For the infinite series

$$1 + 2^1 + 2^2 + 2^3 + 2^4 + \ldots,$$

a natural generalization is a family of geometric progression series

$$1 + q + q^2 + q^3 + \ldots$$

corresponding to different values of the parameter q:

- For $q = 2$, we get the desired series.
- For other values of q, we get different infinite series.

When $|q| < 1$, we can get an explicit expression for the sum. In particular, when $|q| < 1$, the sum of the geometry progression series has a well-defined value: the limit of the values

$$1 + q + \ldots + q^n$$

when n tends to infinity.

To find this limit, we can take into account that if we multiply the sum

$$s_n \stackrel{\text{def}}{=} 1 + q + \ldots + q^n$$

by q, we get

$$q \cdot s_n = q + q^2 + \ldots + q^n + q^{n+1}.$$

So, if we subtract $q \cdot s_n$ from s_n, all the terms cancel out except for 1 and q^{n+1}, and so we get

$$s_n - q \cdot s_n = 1 - q^{n+1},$$

from which we can conclude that

$$s_n = \frac{1 - q^{n+1}}{1 - q}.$$

When n tends to infinity, then q^{n+1} tends to 0, so the expression for s_n tends to

$$\frac{1}{1 - q}.$$

Thus, we get

$$s(q) \stackrel{\text{def}}{=} 1 + q + q^2 + q^3 + q^4 + \ldots = \frac{1}{1 - q}.$$

For different values of the parameter q, we get different values of this sum. In other words, the sum $s(q)$ is a function of q.

How can we extend this expression to the general case? Strictly speaking, the left-hand side sum is defined only when $|q| < 1$, because otherwise, the series diverge. However, the expression $s(q) = \frac{1}{1 - q}$ is an analytical function of the parameter – in the sense that it can be expanded in the power series.

By using these power series, we can naturally define the values of this function $s(q) = \frac{1}{1 - q}$ for the values q for which $|q| \geq 1$.

The only value of the parameter q for which we cannot produce any finite value for $s(q)$ is is the value $q = 1$, because in this case, we have $1 - q = 0$ and thus, the expression $s(q) = \frac{1}{1 - q}$ is infinite.

Thus, for these q, we can *define* the sum of the corresponding infinite series as the value of the function

$$s(q) = \frac{1}{1 - q};$$

see, e.g., [4].

The resulting sum for the simple example with which we started. In particular, for $q = 2$, we get

$$s(2) = \frac{1}{1 - 2} = -1.$$

Thus, according to to the above definition, we have

$$1 + 2^1 + 2^2 + 2^3 + 2^4 + \ldots =$$

$$1 + 2 + 4 + 8 + 16 + \ldots = -1.$$

III. Can We Use This Idea to Compute the Sum of All Natural Numbers: First Attempt

Idea. How can we apply the general idea to the computation of the sum of all natural numbers?

The above geometric progression series lead to the following natural idea: let us consider the sum

$$1 \cdot q^1 + 2 \cdot q^2 + \ldots + n \cdot q^n + \ldots$$

This series converges for $|q| < 1$, and for $q = 1$, we get exactly the desired sum of all natural numbers. So maybe this is a way that we can get the desired sum?

Let us get an explicit expression for the sum when $|q| < 1$. Similarly to the case of the geometric progression, when $|q| < 1$, the sum of the geometry progression series has a well-defined value: the limit of the values

$$1 \cdot q + 2 \cdot q^2 + \ldots + n \cdot q^n$$

when n tends to infinity.

To find this limit, we can take into account that if we multiply the sum

$$s_n \stackrel{\text{def}}{=} 1 \cdot q + 2 \cdot q^2 + \ldots + k \cdot q^k + \ldots + n \cdot q^n$$

by q, we get

$$q \cdot s_n = 1 \cdot q^2 + 2 \cdot q^3 + \ldots + (k-1) \cdot q^k + k \cdot q^{k+1} +$$

$$\ldots + (n-1) \cdot q^n + n \cdot q^{n+1}.$$

So, if we subtract $q \cdot s_n$ from s_n, then, for each power q^k with $k \leq n$, we get the coefficient $k - (k-1) = 1$, and we also get the term $-n \cdot q^{n+1}$:

$$s_n - q \cdot s_n = q + q^2 + \ldots + q^n - n \cdot q^{n+1},$$

from which we can conclude that

$$s_n = \frac{1}{1 - q} \cdot \left(q + q^2 + \ldots + q^n - n \cdot q^{n+1}\right),$$

i.e., that

$$s_n = \frac{q}{1 - q} \cdot (1 + q + \ldots + q^{n-1}) - \frac{n \cdot q^{n+1}}{1 - q}.$$

When n tends to infinity, then $n \cdot q^{n+1}$ tends to 0, and the geometric progression

$$1 + q + \ldots + q^{n-1}$$

tends to

$$\frac{1}{1 - q}.$$

Thus, we get

$$s(q) \stackrel{\text{def}}{=} 1 \cdot q + 2 \cdot q^2 + 3 \cdot q^3 + 4 \cdot q^4 + \ldots = \frac{q}{(1-q)^2}.$$

For different values of the parameter q, we get different values of this sum. In other words, the sum $s(q)$ is a function of q.

We can extend this expression to the general case. Similarly to the case of the geometric progression, we can extend the above formula to other values of q. In particular, for $q = 2$, we have

$$s(2) = \frac{2}{(1-2)^2} = 2,$$

and thus, get yet another finite sum for the infinite series that tends to infinity:

$$1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \ldots = 2.$$

This idea does not work for the sum of all natural numbers. There is only one value q for which this idea does not work, and this is exactly the case $q = 1$ in which we are interested.

So, to compute the sum of all natural numbers, we need to come up with another general family containing this sum.

IV. Computing the Sum of All Natural Numbers: Second Attempt and the Result

How to apply the above idea to the sum of natural numbers: need for an alternative expression. We want to find a general family:

- that would contain the desired infinite series

  $$1 + 2 + 3 + 4 + \ldots$$

  as a particular case – and
- for which we would be able to extend the analytical expression to this case to get a finite answer.

To get a convergent series, a natural idea is to raise things to the power, this will often guarantee convergence. In the previous section, we have tried multiplying natural numbers by powers of some *other number*, this did not work out. A natural idea is then to consider the powers of the natural numbers *themselves*. In other words, let us consider the following family of infinite series:

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots$$

For different values of the parameter s, we get different infinite series. In particular, for $s = -1$, each term

$$\frac{1}{n^s}$$

takes the form

$$\frac{1}{n^{-1}} = \frac{1}{1/n} = n,$$

and the above sum

$$1 + \frac{1}{2^s} + \ldots$$

thus becomes the desired sum

$$1 + 2 + \ldots$$

Estimating the sum for different values of s. The general series converges for $s > 1$; the sum of this series is known as the Riemann zeta function $\zeta(s)$; see, e.g., [1], [3], [5], [6], [9], [10], [11].

Comment. Computation of the values of this function is not as straightforward as the above computation of the sum of the geometric progression, but for some values s, the corresponding result can be rather easily explained. As an example, let us show how to compute the value $\zeta(2)$.

This computation is related to the fact that a polynomial

$$P(x) = x^n + a_{n-1} \cdot x^{n-1} + \ldots + a_1 \cdot x + a_0$$

of n-th degree with n different roots $x_1, \ldots, x_n$ can be represented as the product

$$P(x) = (x - x_1) \cdot \ldots \cdot (x - x_n).$$

This expression is tailored towards polynomials in which the coefficient at the highest degree is 1. We can easily come up with the form tailored towards the polynomials in which the constant is 1, by dividing both sides of the above equality by the product $x_1 \cdot \ldots \cdot x_n$. As a result, we get the expression

$$1 + a_1 \cdot x + a_2 \cdot x^2 + \ldots + a_n \cdot x^n =$$
$$\left(1 - \frac{x}{x_1}\right) \cdot \left(1 - \frac{x}{x_2}\right) \cdot \ldots \cdot \left(1 - \frac{x}{x_n}\right).$$

It is reasonable to expect that for limits of polynomials – i.e., for analytical functions – we get a similar formula

$$1 + a_1 \cdot x + a_2 \cdot x^2 + \ldots + a_n \cdot x^n + \ldots =$$
$$\left(1 - \frac{x}{x_1}\right) \cdot \left(1 - \frac{x}{x_2}\right) \cdot \ldots \cdot \left(1 - \frac{x}{x_n}\right) \cdot \ldots$$

As an example, let us consider the function

$$\frac{\sin(x)}{x}.$$

We know how to expand $\sin(x)$ into Taylor series:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots,$$

so

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots$$

We know all the zeros of this ratio function – they are the value $\pm n \cdot \pi$ corresponding to positive natural numbers n. Thus,

$$\frac{\sin(x)}{x} =$$
$$\left(1 - \frac{x}{\pi}\right) \cdot \left(1 + \frac{x}{\pi}\right) \cdot \left(1 - \frac{x}{2 \cdot \pi}\right) \cdot \left(1 + \frac{x}{2 \cdot \pi}\right) \cdot \ldots$$

For each pair of roots $\pm n \cdot \pi$, we get

$$\left(1 - \frac{x}{n \cdot \pi}\right) \cdot \left(1 + \frac{x}{n \cdot \pi}\right) = \left(1 - \frac{x^2}{n^2 \cdot \pi^2}\right).$$

Thus,

$$\frac{\sin(x)}{x} =$$

$$\left(1 - \frac{x^2}{\pi^2}\right) \cdot \left(1 - \frac{x^2}{2^2 \cdot \pi^2}\right) \cdot \ldots \cdot \left(1 - \frac{x^2}{n^2 \cdot \pi}\right) \cdot \ldots$$

When we multiply all these polynomials, we can take into account that the only terms proportional to x^2 are terms coming from the case when all multiplied coefficients are 1s except for one of them – else we would have a term proportional to x^4 or x^6, etc. In general, we have

$$(1 + a_1 \cdot x^2) \cdot (1 + a_2 \cdot x^2) \cdot \ldots =$$

$$1 + (a_1 + a_2 + \ldots) \cdot x^2 + \ldots,$$

i.e., in our case,

$$\frac{\sin(x)}{x} =$$

$$x^2 \cdot \left(\frac{1}{\pi^2} + \frac{1}{2^2 \cdot \pi^2} + \ldots + \frac{1}{n^2 \cdot \pi^2} + \ldots\right) + \ldots$$

On the other hand, we know that the coefficient at x^2 is the Taylor expansion of the ration function is equal to

$$-\frac{1}{3!} = -\frac{1}{1 \cdot 2 \cdot 3} = -\frac{1}{6},$$

thus,

$$\frac{1}{6} = \frac{1}{\pi^2} + \frac{1}{2^2 \cdot \pi^2} + \ldots + \frac{1}{n^2 \cdot \pi^2} + \ldots$$

Multiplying both sides of this equality by π^2, we conclude that

$$\zeta(2) \stackrel{\text{def}}{=} 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} + \ldots = \frac{\pi^2}{6}.$$

By comparing coefficients at x^4, x^6, etc., we can similarly compute the values $\zeta(4)$, $\zeta(6)$, etc.

Extending this function to different values s. The function $\zeta(s)$ is analytical, and it can be extended to values $s \le 1$.

It is interesting to mention that this extension was first discovered by B. Riemann in the 19th century [9]. Riemann actually extended this function not just to real values $s \le 1$, but also to *complex* values of s. Riemann showed that $\zeta(s) = 0$ for $s = -2$, $z = -4$, $z = -6$, etc. He also noticed – but could not prove it – that all other zeros of this function have real part equal to 0.5. This hypothesis of Riemann leads to interesting consequences about prime numbers – but until now, no one has been able to prove or disprove this hypothesis.

Riemann's hypothesis is probably the most well-known open problem in mathematics [1], [2], [3], [5], [6], [9], [10], [11].

In this paper, we are interested in the value $s = -1$. For this s, the resulting analytical function $\zeta(s)$ attains the value

$$\zeta(-1) = -\frac{1}{12}.$$

This is the meaning behind the above equality

$$1 + 2 + 3 + 4 + \ldots = -\frac{1}{12}.$$

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