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# HOW TO EXPLAIN THE EMPIRICAL SUCCESS OF GENERALIZED TRIGONOMETRIC FUNCTIONS IN PROCESSING DISCONTINUOUS SIGNALS

**Fourier series and their limitations: a brief reminder.** One of the discoveries of Isaac Newton was that if we place a prism in the path of (white) solar light, this light will decompose into lights of different colors. From the mathematical viewpoint, a monochromatic light is a sinusoid, i.e., the dependence $x(t)$ of its intensity $x$ on time $t$ has the form $x(t) = A \cdot \sin(\omega \cdot t + \varphi)$ for some constants $A$, $\omega$, and $\varphi$. The intensity of original white light is equal to the sum of these components, i.e., to $x(t) = \sum\limits_{i=1}^{n} A_i \cdot \sin(\omega_i \cdot t + \varphi_i)$.

Newton showed that any light can be decomposed in this way, i.e., in effect, that any signal $x(t)$ can be represented as a linear combination of sinusoids corresponding to different frequencies $\omega$.

This idea was explored in the early 19 century by Jean-Baptiste Joseph Fourier, who showed that this representation helps in solving many physics-related differential equations. Computational methods based on such a representation are known as Fourier techniques. At present, these techniques are ubiquitous in science and engineering; see, e.g., [5].

However, the Fourier techniques have their limitations: while they works well for smooth signals, they do not work as well for discontinuous signals that describe abrupt transitions – such as phase transitions, earthquakes, etc. Specifically, if we represent a discontinuous signal as a sum of sinusoids, we get large oscillations near the discontinuity; this is known as the *Gibbs phenomenon.*

It is possible to avoid these oscillations if, instead of representing a signal as a linear combination of sinusoids, we represent it as a linear combination of discontinuous functions – e..g., Haar wavelets [4] – but the resulting representation is not very computationally efficient for smooth signals.

It is therefore desirable to come up with a representation which would be efficient both for smooth and for discontinuous signals.

**Generalized trigonometric functions.** A successful semi-heuristic approach to solving the above problem is the use of *generalized trigonometric functions* instead of the sinusoids. Specifically, a sinusoid can be defined as a function which is inverse to the integral

$$\int \frac{dt}{\sqrt{1-t^2}} = \int \frac{dt}{(1-t^2)^{1/2}},$$

expanded by periodicity to the entire real line. A generalized trigonometric function can be defined as a periodic extension of an inverse function to a more general integral

$$\int \frac{dt}{(1-t^p)^{1/q}}$$

for general values $p$ and $q$. The derivative of this generalized function is no longer everywhere continuous – and the farther $p$ and $q$ from the value 2, the larger this discontinuity.

Empirically, these functions – for appropriate $p$ and $q$ – are good approximations both for smooth and for discontinuous signals; see, e.g., [2, 3].

**Challenge.** The empirical success is here, but so far, there have been no convincing theoretical explanation for this success. In principle, we can think of many generalizations of trigonometric functions, and it is not clear whey namely this generalization is empirically successful.

This absence of theoretical explanation prevents the wider use of this technique: the users are reluctant to use it, since they are not sure that the empirical success so far is not an artifact.

**Our objective.** In this paper, we provide a physics-motivated theoretical explanation for the empirical success of the generalized trigonometric functions.

**Physical meaning of sinusoids: reminder.** Sinusoidal signals are frequently observed in nature, because they correspond to simple oscillations. Namely, they correspond to situations in which the potential energy $E_{\rm pot}$ is equal to $E_{\rm pot}=\frac{1}{2}\cdot c\cdot x^2$ for some constant $c$. In Newtonian mechanics, the kinetic energy is equal to $E_{\rm kin}=\frac{1}{2}\cdot m\cdot(\dot x)^2$. Thus, the overall energy $E = E_{\text{pot}} + E_{\text{kin}}$ is equal to

$$E = \frac{1}{2} \cdot c \cdot x^2 + \frac{1}{2} \cdot m \cdot (\dot{x})^2.$$

Sinusoidal oscillations correspond to the idealized case when we can ignore the friction and when, therefore, the energy is preserved:

$$\frac{1}{2} \cdot c \cdot x^2 + \frac{1}{2} \cdot m \cdot (\dot{x})^2 = E_0 = \text{const.}$$

Thus, once we know the coordinate $x$, we can determine $\dot{x}$ as

$$(\dot{x})^2 = \frac{2E_0 - c \cdot x^2}{m},$$

so

$$\dot{x} = \frac{dx}{dt} = \frac{\sqrt{2E_0 - c \cdot x^2}}{\sqrt{m}}.$$

This equation can be simplified if we separate the variables, i.e., if we move all the terms related to $x$ to the left-hand side and all the terms related to $t$ to the right-hand side. This can be done if we divide both sides of the above formula by the right-hand side and then multiply both sides by $dt$:

$$\sqrt{m} \cdot \frac{dx}{\sqrt{2E_0 - c \cdot x^2}} = dt.$$

In appropriately selected units of time and $x$, we have

$$dt = \frac{dx}{\sqrt{1 - x^2}},$$

thus, the dependence $t(x)$ of $t$ on $x$ has the form

$$t = \int \frac{dx}{\sqrt{1 - x^2}}.$$

The desired dependence $x(t)$ of $x$ on $t$ is the inverse function – which, as we have mentioned, is exactly the sinusoid.

**Discussion.** The formula for the potential energy $E_{\text{pot}} = \frac{1}{2} \cdot c \cdot x^2$ is *scale-invariant* in the sense that:

- if we change the measuring unit for $x$ to a one which is $\lambda$ times smaller and thus, change all the numerical values from $x$ to $x' = \lambda \cdot x$,

- then, by appropriately re-scaling the unit for measuring energy, i.e., by taking $E' = \lambda^2 \cdot E$, we will have the exact same dependence between $E'$ and $x'$ in the new units: $E' = \frac{1}{2} \cdot c \cdot (x')^2$.

Similarly, the dependence $E_{\rm kin} = \frac{1}{2} \cdot c \cdot (\dot{x})^2$ is also scale-invariant.

**Our idea.** Scale-invariance – i.e., the fact that the physical laws do not depend on the choice of measuring units – is an important physical principle. However, scale-invariance does not necessarily mean that the potential energy should be proportional to the square of $x$: e.g., the dependence $E_{\rm pot} = x^3$ is also scale-invariant.

Let us therefore consider a general case in which both components $E_{\rm pot}(x)$ and $E_{\rm kin}(\dot{x})$ of the overall energy $E = E_{\rm pot}(x) + E_{\rm kin}(\dot{x})$ are scale-invariant.

**Our idea leads exactly to generalized trigonometric functions.** Scale-invariance of the dependence $E_{\rm pot}(x)$ means that for every parameter $\lambda$ describing re-scaling of the coordinate $x$, there exists an appropriate re-scaling $\mu(\lambda)$ of energy that preserves this dependence, i.e., for which $E = E_{\rm pot}(x)$ implies that $E' = E_{\rm pot}(x')$, where $E' = \mu(\lambda) \cdot E$ and $x' = \lambda \cdot x$. Substituting the expressions for $E'$ and $x'$ into the above formula, we get $\mu(\lambda) \cdot E = E_{\rm pot}(\lambda \cdot x)$. Since $E = E_{\rm pot}(x)$, we thus get $\mu(\lambda) \cdot E_{\rm pot}(x) = E_{\rm pot}(\lambda \cdot x)$.

It is known (see, e.g., [1]) that all continuous (or even integrable) solutions of this functional equation have the form $E_{\rm pot}(x) = c \cdot x^p$ for some constants $c$ and $p$. Similarly, scale-invariance of the expression for kinetic energy implies that $E_{\rm kin}(\dot{x}) = m \cdot (\dot{x})^q$ for some constants $m$ and $q$.

Thus, the overall energy $E = E_{\rm kin} + E_{\rm pot}$ takes the form $E = c \cdot x^p + m \cdot (\dot{x})^q$. In the no-friction approximation, energy is preserved, so the left-hand side is a constant. By selecting appropriate units for energy, we can make this constant equal to 1. Then, by selecting appropriate units for $x$ and for time (hence for $\dot{x}$), we can get a simplified expression $1 = x^p + (\dot{x})^q$. In this case, $(\dot{x})^q = 1 - x^p$, hence

$$\dot{x} = \frac{dx}{dt} = (1 - x^p)^{1/q},$$

so

$$dt = \frac{dx}{(1 - x^p)^{1/q}}$$

and

$$t(x) = \int \frac{dx}{(1 - x^p)^{1/q}}.$$

The desired dependence $x(t)$ is the inverse function to this integral $t(x)$ – and is, thus, exactly the above-described generalized trigonometric function.

**Conclusion.** We have shown that a seemingly arbitrary generalization of sinusoids can be naturally derived from a physically meaningful model – and the only functions obtained from this model are indeed the generalized trigonometric functions. This derivation provides a theoretical explanation of the empirical success of these functions – while there are many mathematically possible generalizations of sinusoids, these functions are the only one which are consistent with the corresponding physical model.

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