6th Asian-Pacific Symposium on Structural Reliability and its Applications (APSSRA6)
28-30 May 2016, Shanghai, China
H.W. Huang, J. Li, J. Zhang & J.B. Chen (editors)

Do we have compatible concepts of epistemic uncertainty ?

M. Beer[1,2,3], S. Ferson[4] and V. Kreinovich[5]

[1]*Leibniz University Hannover, Hannover, Germany. Email: beer@bauinf.uni-hannover.de*
[2]*University of Liverpool, Liverpool, UK. Email: mbeer@liverpool.ac.uk*
[3]*Tongji University, Shanghai, China*
[4]*Applied Biomathematics, Setauket, NY USA. Email: scott@ramas.com*
[5]*University of Texas at El Paso, El Paso, TX, USA. Email: vladik@utep.edu*

Abstract: Epistemic uncertainties appear widely in civil engineering practice. There is a clear consensus that these epistemic uncertainties need to be taken into account for a realistic assessment of the performance and reliability of our structures and systems. However, there is no clearly defined procedure to meet this challenge. In this paper we discuss the phenomena that involve epistemic uncertainties in relation to modeling options. Particular attention is paid to set-theoretical approaches and imprecise probabilities. The respective concepts are categorized, and relationships are highlighted.

Keywords: Epistemic Uncertainties, Subjective Probabilities, Imprecise Probabilities, Probability Boxes, Fuzzy Probabilities

1. Introduction

The modeling of epistemic uncertainties is a challenging task, not only because of diversity and limitation of the available information but also because of the variety of concepts and approaches, from which the engineer can chose. This choice is made difficult by the perception that the available concepts are competing and opposed to one another rather than being complementary and compatible. Clearly, the first consideration should be devoted to probabilistic modelling, naturally through subjective probabilities, which express a belief of the expert and can be integrated into a fully probabilistic framework in a coherent manner via a Bayesian approach. While this pathway is widely accepted and recognized as being very powerful, the potential of set-theoretical approaches and imprecise probabilities has only been utilized to some minor extent. Those approaches, however, attract increasing attention in cases when available information is not rich enough to specify subjective probability distributions (Beer et al. 2013). In this paper we elucidate aspects of epistemic uncertainties in relation to the variety of models for uncertainty and imprecision. The main features of these concepts and their inter-relations are discussed. Attention is devoted to an explanation of their relationship and compatibility to probabilistic approaches.

2. Epistemic Uncertainty

Epistemic uncertainty is generally understood as the analyst's lack of knowledge and thus reducible by gathering additional information. Unfortunately, this understanding does not imply a specific mathematical model. It rather summarizes a variety of problematic cases that are commonly encountered in the modeling of uncertainties in civil engineering practice. An analysis of these situations, however, reveals that specific models can be chosen based on the features of the epistemic uncertainty in each particular case.

If the reason for epistemic uncertainty is subjectivity, subjective probabilities provide a suitable framework for modeling, which is consistent with the axioms of probability theory. In this context Bayesian approaches have become very popular, they allow to use expert knowledge to compensate for the paucity of data and limitations of information. If a subjective perception regarding a probabilistic model exists and some data for a model update can be made available, a Bayesian approach can be very powerful, and meaningful results using available information can be derived. Bayesian approaches are attracting increasing attention in engineering. Considerable advancements have been reported for the solution of various engineering problems, in particular, in the area of system identification and model updating, see for example (Au 2012a,b) and (Mottershead et al. 2011). Here, one can usually build on a reasonable basis of expert knowledge to specify a suitable model class and to cast prior knowledge into subjective distribution functions. If subjective probabilistic statements can be formulated on rational grounds and some data of suitable quality are available, then Bayesian updating can play its important role. The subjective influence in the model assumption decreases quickly when the amount of data increases. When data are available for such updating, a probabilistic model parameter can be estimated with the expected value of a posterior distribution. In this case the result is a mix of objective and subjective information. Alternatively, the epistemic uncertainty represented by the posterior distribution can be made visible in the result, for example in form of credible intervals, which can be helpful for the communication of the results, as explained in (Ellingwood 2009) in the context of risk assessment. This treatment of subjective information enables taking into account epistemic uncertainty together with aleatory uncertainty (stochastic variability) in a probabilistic framework.

Epistemic uncertainty, however, is not limited to subjectivity but may also refer to indeterminacy, ambiguity, fragmentary or dubious information and other phenomena, for which there seems to be no natural description in terms of subjective probabilities. Examples are poor data or linguistic expressions, which indicate a possible value range or bounds rather than a subjective distribution function. In the early design stage, design parameters can be specified only roughly and underlie later changes as the design matures. Physical inequalities can frequently be utilized to determine bounds for parameters but not to specify characteristics concerning variations, fluctuations, value frequencies, etc. over some value range. The same applies to the numerical description of individual measurements obtained under dubious conditions. Conditional probabilities determined under unknown conditions and marginals of a joint distribution with unknown copula (dependence function) provide bounds for probabilistic models rather than prior probabilistic information for model options. This facet of epistemic uncertainty is associated with several different manifestations of an uncertain variable:

(i) the variable may take on any value between bounds, but there is no basis to assume probabilities to the options;

(ii) the variable has a particular real value, but that value is unknown except that it is between bounds;

(iii) the variable may take a single value or multiple values in some range, but it is not know which is the case;

(iv) the variable is set-valued.

The characteristics of this type of information can be described most appropriately as imprecision. Mathematical models proposed for imprecise variables are set-theoretical and include intervals, Bayesian, rough sets, clouds, and convex models. Overviews on respective applications in engineering can be found in (Möller and Beer 2008) and (Moens and Vandepitte 2005).

When epistemic uncertainty appears as imprecision, a subjective probabilistic model description would be quite arbitrary. Consider a floor beam with a strict requirement for the maximum deflection. Suppose the dependency between load and deflection is known deterministically, but a load parameter is available in the form of bounds only. This information is naturally modeled as an interval. Since no information about any probabilities exists, one could now assign a uniform distribution to the load interval based on the principle of maximum entropy. This approach is perhaps reasonable in the context of information theory, but it is disconnected from the engineering context of the problem itself. It leads to an averaged result for the deflection of the beam using equal weights for all possible load values within the available interval. However, the maximum deflection, which is of interest, is not directly addressed and can only be retrieved from simulation results with tremendous effort. And for another assigned probability distribution over the load interval the result would be different. Thus the character of the available information is changed; the interval input is transformed into a probabilistic result, the meaning of which is based purely on subjective or really arbitrary assumptions and justifications, which may even be out of context. In contrast to this, an interval analysis ensures a consistent translation of the input interval into a result interval without asking for any subjective assumptions. The character of the available information is retained in this analysis. And it delivers directly the maximum deflection, which is the quantity of interest, as bounds on the quantity. This simple example shows how important an appropriate modeling of epistemic is to not undermine the purpose of an analysis, potentially with severe consequences.

The modeling of imprecision is not limited to the use of intervals. An interval is a quite crude expression of imprecision. The specification of an interval for a parameter implies that, although a number's value is not known exactly, exact bounds on the number can be provided. This is not always realistic because the task of specifying precise numbers is just transferred to the bounds. Fuzzy set theory provides a workable basis for relaxing the need for precise values or bounds. It allows the specification of a smooth transition for elements from belonging to a set to not belonging to a set. Fuzzy numbers are a generalization and refinement of intervals for representing imprecise parameters and quantities. The essence of an approach that uses fuzzy numbers that distinguishes it from more traditional approaches is that it does not require the analyst to circumscribe the imprecision all in one fell swoop with finite characterizations having known bounds. The analyst can now express the available information in the form of a series of plausible intervals, the bounds of which may grow, possibly even to infinite limits. This allows a more nuanced approach compared to interval modeling. Fuzzy sets provide an extension to interval modeling that considers variants of interval models, in a nested fashion, in one analysis. This modeling of imprecision is analogous to probability's modeling of uncertainty, and, like the probabilistic approach, it also produces a distributional answer that is more nuanced than what can be achieved by worst case analysis or bounding with a simple interval. Fuzziness arises in cases where there are degrees or gradations admitting arbitrariness in where defining lines are drawn. In other fields, this is sometimes called vagueness.

3. Imprecise Probabilities

The distinction between probabilistic subjectivity and imprecision as different forms of epistemic uncertainty provides a pragmatic criterion for classifying non-deterministic phenomena according to the nature of information. From this perspective, aleatory uncertainty (stochastic variation) and the subjective probabilistic form of epistemic uncertainty can be summarized as probabilistic uncertainty, whereas imprecision refers to the non-probabilistic form of epistemic uncertainty. This classification helps to avoid confusion if uncertainty appears as both probabilistic and non-probabilistic phenomena simultaneously in an analysis. An illustrative example for this situation is a random sample of imprecise perceptions (e.g., intervals due to limited measurement accuracy) of a physical quantity. While the scatter of the realizations of the physical quantity possesses a probabilistic character (frequentist or subjective), each particular realization from the population exhibits, additionally, imprecision with a non-probabilistic character. If an analysis involves this type of hybrid information, it is imperative to consider imprecision and probabilistic uncertainty simultaneously but not to mix the characteristics, so that imprecision is not described in terms of a probabilistic model and vice versa.

This conceptual understanding and classification into probabilistic uncertainty and imprecision provides intuitive motivation for imprecise probabilities and their terminology. This concept to deal with imprecision and uncertainty simultaneously in the same problem is conceptually different from a subjective probabilities approach. Suppose only bounds on some parameter of a distribution are known. Any appropriate distribution whose parameter is limited to these bounds might then be considered an option for modeling. But the selection of any particular distribution would introduce unwarranted information that cannot be justified except by bold assumption. Even assuming a uniform distribution, which is commonly done in such cases, ascribes more information than is actually given by the bounds. This situation may become critical if no data (or only very limited amount of data) is available for a Bayesian update. The initial subjectivity is then dominant in the posterior distribution and in the final result. If these results, such as failure probabilities, determine critical decisions, one may wish to consider the problem from the following perspective.

When several probabilistic models are plausible for the description of a problem, and sufficient information is not available to assess the suitability of the individual models or to relate their suitability with respect to one another, then it may be of interest to identify the *range* of possible outcomes, including especially the worst possible case, rather than to average over all plausible model options with arbitrary weighting. The probabilistic analysis is carried out conditionally on each of many particular probabilistic models out of the set of plausible models. In reliability assessment, this implies the calculation of an upper bound for the failure probability as the worst case. This perspective can be extended to explore the sensitivity of results with respect to the variety of plausible models, that is, with respect to a subjective model choice.

A mathematical framework for an analysis of this type has been established with imprecise probabilities. But this intuitive view is by no means the entire motivation for imprecise probabilities. Imprecise probabilities are not limited to a consideration of imprecise distribution parameters. They are also capable of dealing with imprecise conditions, with dependencies between random variables, and with imprecise structural parameters and model descriptions (Klir 2006, Walley 1991). Further, multivariate models and statistical estimations and tests with imprecise sample elements can be constructed, results from robust statistics in the form of solution domains of statistical estimators can be considered directly. Recent overviews on imprecise probabilities with applications in engineering are provided in (Beer et al. 2013) and (Augustin et al. 2014).

A key feature of imprecise probabilities is the identification of bounds on probabilities for events of interest; the uncertainty of an event is characterized with two values; a lower probability and an upper probability. The distance between the lower and upper probability bounds reflects the indeterminacy in model specifications expressed as imprecision of the models. This impression results from not introducing artificial model assumptions. It is described by implementing set-valued descriptors in the specification of a probabilistic model. The model description is thereby limited to an appropriate domain, and no further specific characteristics are ascribed. This introduces significantly less information in comparison with a specific subjective distribution function as used in a Bayesian approach. Imprecision in the model description expressed in a set-theoretical form is not translated into probabilities; it is not described in terms of probabilities, instead, it is reflected in the result as a *set* of probabilities which covers all plausible cases of model assumptions. This feature is particularly important when the calculated probabilities provide the basis for critical decisions.

With imprecise probabilities the analysis may be performed with various relevant models to obtain a set of relevant results and associated decisions. This helps to avoid wrong decisions due to artificial restrictions in modeling.

In the first systematic discussion of imprecise probabilities (Walley 1991) their semantics is summarized with the term "indeterminacy", which arises from ignorance about facts, events, or dependencies. This specifies the context in which imprecise probabilities appear in nature and shows a basic difference from Bayesian and traditional probabilistic analysis. In view of engineering problems imprecise probabilities arise, in particular, when probabilistic elicitation exercises are incomplete, when probabilistic information appears incomplete or dubious, and when observations of sample elements appear imprecise. Further motivations for imprecise probabilities include observations which cannot be separated clearly, conditional probabilities which are observed with unclear conditions, and marginals of a distribution on a joint space which are specified with imperfect information about the accompanying copula function that characterizes the dependence among the variables.

4. Conceptual Categorization

The imprecise probability approach includes a large variety of specific theories and mathematical models associated with an entire class of measures. This variety is discussed in (Klir 2006) in a unifying context; the diversity of model choices is highlighted, and arguments for imprecise probabilities are summarized. Imprecise probabilities have a close relationship to the theory of random sets and cover, for example, the concept of upper and lower probabilities, sets of probability measures, distribution envelopes, probability bounds analysis using p-boxes, interval probabilities, Choquet capacities of various orders, and evidence theory as a theory of infinitely monotone Choquet capacities. Moreover, fuzzy probabilities, with their roots in the theory of fuzzy random variables, are also covered under the framework of imprecise probabilities and have strong ties to several of the aforementioned concepts.

The ideas of imprecise probabilities may be categorized into three basic groups of concepts associated with three different technical approaches to construct imprecise probabilistic models.

1) Events, which may be complex, are observed phenomenologically and are recorded with coarse specifications. Such a specification might be, for example, "severe shear cracks in a wall"'. In general, these coarse specifications may be the best information available, or they may arise from limitations in measurement feasibility. The latter applies, for example, to damping coefficients. There is typically no probabilistic information available to specify distribution functions for these coarse specifications, so that modeling of indeterminacy as sets is most appropriate. An expert may then assign probabilities to entire sets, which represent the observations. Starting from this model, bounds for a set of distribution functions can be constructed. This view is followed in evidence theory.

2) Parameters of a probabilistic model, the distribution type or, in a non-parametric description, the curve of the cumulative distribution function may only be specified within some bounds. This imprecision may arise, for example, when conflicting information regarding the distribution type is obtained from statistical tests, that is, when the test results for different distributions as well as for compound distributions thereof with any mixing ratio are similar. These test results do not provide grounds for assigning probabilities to the model options. If no additional information is available in such situations, the most suitable approach for modeling the corresponding uncertainty is as a set of distributions. In the simplest form, this implies the use of intervals for the distribution parameters. The concept of interval probabilities follows this idea.

3) Outcomes from a random experiment may appear as blurred, for example, due to limitations in the measurement feasibility or due to the manner of characterization of the outcomes. This characterization can emerge, for example, in form of linguistic variables such as when asking a group of people for their perception of the temperature in a room, the results appear as "warm", "comfortable"', "slightly warm" etc. This type of information is typically described by fuzzy sets, which provide in contrast to traditional sets the additional feature of a membership function. The membership function for an individual observation, in this context, does not represent any probabilistic information; it expresses a degree of truth with which certain numerical values represent the characterization of the observation (for example, the statement "warm"). It also provides a tool for a more nuanced investigation with respect to the magnitude of imprecision. The imprecise perception of a random variable can be translated into a traditional set or fuzzy set of distribution functions. This concept complies with the model of fuzzy random variables.

Although some concepts of imprecise probabilities do not completely fall into one of these groups, they usually show clear relationships to them and can be constructed out of them or as combinations thereof. There are also strong relationships between the groups. A common feature of all concepts of imprecise probabilities is the consideration of an entire set of probabilistic models in one analysis. In the results, bounds on probabilities for events of interest are calculated. The theoretical differences between the concepts mainly concern the mathematical description of the set of probabilistic models and the connection to the probabilistic models involved. Thus, from a practical point of view, this categorization and the associated features of the concepts can provide the engineer with a good sense for the modeling of a problem. In any case, the choice of the concept should be driven by both the nature of the available information and the purpose of the analysis.

5. Probability Boxes and Fuzzy Probabilities

Probability boxes and fuzzy probabilities cover all three groups of concepts as described above. These two concepts are closely related to one another since fuzzy probabilities can be considered as nested probability boxes and probability boxes can be viewed as degenerated case of fuzzy probabilities.

Probability bounds analysis has been established in (Ferson and Hajagos 2004) and (Ferson et al. 2003). If evidence theory is based on the idea that physical values can be bounded rather than specified as points, and interval probability is based on the idea that probabilities can be bounded rather than necessarily given as point values, then probability bounds analysis is based on the combination of these dual ideas. It is a numerical approach that allows the calculation of bounds on arithmetic combinations of probability distributions when perhaps only bounds on the input distributions are known. These bounds are called probability boxes, or p-boxes, and constrain cumulative probability distributions (rather than densities or mass functions). This bounding approach permits analysts to make calculations without requiring overly precise assumptions about parameter values, dependence among variables, or distribution shapes. In principle, the approach allows the analyst to decide what assumptions are reasonable and what are not. When the information about a distribution is very good, the bounds on the distribution will be very tight, approximating the precise distribution that is used in traditional probabilistic approaches. When the information is very poor, the bounds will tend to be much wider, representing weaker confidence about the specification of this distribution.

Probability bounds analysis is essentially a unification of interval analysis with traditional probability theory. It gives the same answer as interval analysis does when only range information is available. It also gives the same answers as a traditional probabilistic approach does when information is abundant enough to precisely specify input distributions and their dependencies. Thus, it is faithful to both theories and generalizes them to solve problems neither could solve alone. Probability theory has facilities for modeling correlations and dependencies, but cannot easily distinguish between variability and ignorance. Interval analysis expresses ignorance, but it has no useful notions of central tendency or moments and it cannot easily handle dependence among variables. Probability bounds analysis incorporates techniques from probability theory for modeling correlations and dependencies and projecting distribution moments through mathematical expressions. From interval analysis, it inherits its fundamental conception of set-valued epistemic uncertainty, as well as important ancillary computational techniques.

The diverse methods comprising probability bounds analysis provide algorithms to evaluate mathematical expressions when there is uncertainty about the input values, their dependencies, or even the form of mathematical expression itself. The calculations yield results that are guaranteed to enclose all possible distributions of the output variable as long as the input p-boxes were all guaranteed to enclose their respective distributions. In some cases, a calculated p-box will also be best-possible in the sense that the bounds could be no tighter without excluding some of the possible distributions. As a bounding approach, probability bounds analysis can effectively propagate some kinds of uncertainties that cannot be comprehensively addressed by any traditional probabilistic approach, even with sampling and in theory with infinitely many samples. For instance, if an analyst does not know the distribution family for some input, a distribution-free p-box can be used to bound all possible distribution families consistent with the other information available about that variable. Likewise, if the nature of the stochastic dependence between two distributions is unknown, probability bounds analysis can be used to bound all possible distributions that might arise as a function of the inputs whatever their interdependence might be. Such calculations are not possible with a traditional probabilistic approach combined with a sensitivity analysis involving multiple Monte Carlo simulations, because such problems are intrinsically infinite-dimensional.

Fuzzy probability theory, see (Beer 2009) and (Buckley 2005), can be regarded as a marriage between fuzzy set theory and probability theory. It enables the consideration of a fuzzy set of probabilistic models, which are variously plausible according to the available information. Probabilistic information is captured by probabilistic models, and imprecision in the probabilistic model specification is described by fuzzy sets. This preserves uncertainties as probabilistic information and imprecision as set-theoretical information throughout the entire analysis. In the case when only fuzzy information is available, the special case of a pure fuzzy analysis appears. On the other hand, if all information can be captured with precisely specified probabilistic models, the result becomes equal to the traditional probabilistic result.

With the interpretation of fuzzy sets as an extension of intervals, the very close relationship between fuzzy probabilities and probability boxes becomes obvious. A fuzzy set of probabilistic models can be regarded as a set of probability boxes allowing the consideration of various box sizes in a nested fashion in one analysis. A fuzzy probabilistic model can, hence, be formulated in the same manner as a probability box, but provides the additional nuanced description of the imprecision in the probabilistic model. Interval-valued information in the specification of parameters, distribution types, dependencies, or functional values of a distribution can be implemented including a gradual subjective assessment of the interval sizes. For example, the results from interval estimations on various confidence levels and conflicting statistical test results for various thresholds of rejection probabilities can be used as the basis for a modeling with stepwise changing interval sizes. This perspective relates fuzzy probabilities to interval probabilities, where the imprecision emerges in the probability measure. But it is also connected to evidence theory in the same way as probability boxes. When the focal sets in evidence theory are interval-valued images of random elementary events, so that the basic probability assignment is determined and not a subjective matter left with the analyst, then p-boxes can be constructed by belief and plausibility distributions. When the focal sets appear as fuzzy-valued images of random elementary events, then p-boxes can be obtained in the same way for each α-level of a fuzzy set, leading to a fuzzy probability distribution in overall. Once a fuzzy probabilistic model is established, the same analysis methods as in a p-box approach can be used for processing, namely, they are applied to each α-level. Thus, for any selected α-level, the complete framework of probability bounds analysis is applicable.

In this context, it becomes obvious that the membership function of the corresponding fuzzy sets serves only to summarize various plausible interval models into a single embracing scheme. No interpretation of the meaning of membership values is involved. The importance of fuzzy modeling lies here in the simultaneous consideration of various magnitudes of imprecision at once in the same analysis.

The nuanced features of fuzzy probabilities provide extended insight into engineering problems. A fuzzy probabilistic analysis can be utilized to identify sensitivities of the failure probability with respect to the imprecision in the probabilistic model specification, i.e. with respect to the probabilistic model choice. It can be used for analyses in early design stages, where it can help to reduce imprecision in order to meet the design criteria. Conclusions can be drawn to increase the sample size or to set up quality control measures. Relating the magnitude of fuzziness in the model specifications to the magnitude of fuzziness of the probabilistic results, a robust design can be derived, which is not sensitive with respect to the probabilistic model assumptions, i.e., a design which works for all possible probability distributions which are consistent with our knowledge.

The engineering capabilities of these approaches become obvious in the solution to large scale, complex challenge problems that involve significant epistemic uncertainties (Patelli et al. 2015).

6. Compatibility to Subjective Probabilities

Imprecise probabilities and subjective probabilities follow different conceptual approaches to model epistemic uncertainty. These approaches are associated with different requirements and features. A probabilistic approach requires a subjective perception of the probabilities. Epistemic uncertainty is processed via a weighted average according to the subjective distributions. This minimizes the influence of extremes. Epistemic uncertainty is translated into a result that is most likely in accordance with the initial subjective assessment of the input information. Estimates for an expected value or mode can be derived with a higher confidence compared with estimates for higher moments or tail probabilities. Reduction of epistemic uncertainty through additional information translates here into an increased quality of the estimates for higher moments or tail probabilities. This may be seen as an approach of the results "from the inside out" to the epistemic uncertainty.

In contrast to subjective probabilities, imprecise probabilities involve the quantification of epistemic uncertainty with set-theoretic descriptors. In some cases the specification of sets may be possible in absolute terms, for example, when imprecision of digital measurements is due to a limited number of digits. In other cases sets can be specified in the sense of credibility intervals so that they carry probabilities. Sets with "soft boundaries", such as fuzzy sets, can be utilized to accommodate vagueness in the specification of the boundaries. In any case, no assumption is made about the probabilities of different elements from this set. Instead, each and every element from a set is considered as plausible with no weighting with respect to one another. This is equivalent to a consideration of all possible distributions over the set, including Dirac functions. The probabilistic analysis is carried out conditionally on the elements from the sets, which leads eventually to sets of probabilistic results. The sets of probabilistic results provide bounds on the results (which can be associated with some confidence level). This facilitates a best and worst case analysis within the epistemic uncertainty. Reduction of epistemic uncertainty through additional information translates here into closer bounds on the results. This may be seen as an approach of the results "from the outside in" to the epistemic uncertainty".

From the conceptual philosophies it is clear that the two approaches to epistemic uncertainty provide different avenues but do not compete with one another. The two approaches show complementarity with respect to one another, when there is no clear way to prefer one of the models. In those cases both approaches can be applied in parallel to obtain insight from two perspectives. The results from both approaches can be interpreted in relation with one another. It can be found where the best possible estimate for a result is obtained with respect to the set of all plausible results, perhaps on some confidence level. At the same time, the width of the result bounds makes the magnitude of imprecision visible. These results allow important conclusions on model capabilities.

A respective case study that underlines the above conclusions is described in (Beer et al. 2014). The discussions include a combination of both approaches based on the developments in (Stein et al. 2013), which show another perspective of compatibility. Through combination of imprecise probabilities with subjective probabilities, a Bayesian analysis can be performed with a set of priors with no weighting applied, with imprecise data and with combinations thereof. The research shows that the imprecision in the specification of priors decays with growing sample size as the influence of the priors decays itself. Also, an update with imprecise data converges to the statistical result with imprecise data alone and no subjective prior information. These results underline the consistency of the approaches.

Recent research expands on the idea to develop a unified approach that combines frequentist and Bayesian approaches. For instance, Quick Bayes (Ferson et al. 2014) is an approach for estimating parameters in probabilistic models when the available empirical information about those parameters and the output from the model are scarce or imprecise. Quick Bayes is similar to robust Bayes analysis (Berger 1994, Walley 1991), so it can be thought of as a kind of automated Bayesian sensitivity analysis which accounts for epistemic uncertainty about the prior, the likelihood or both. But Quick Bayes is especially convenient to use because the analyst need not choose a prior distribution when no particular evidence or belief justifies such a choice. More importantly, and unlike traditional Bayesian analysis, Quick Bayes parameter estimates express traditional Neyman confidence for the parameter or result being estimated. The estimators encode confidence intervals at all possible confidence levels all at the same time, so they represent a comprehensive characterization of the inferential uncertainty about the estimate.

Like Bayesian posterior distributions, Quick Bayes estimates can be used in subsequent calculations. Because the Quick Bayes estimators have the form of p-boxes, they can be propagated using the ordinary machinery of probability bounds analysis. Although confidence intervals cannot easily be projected in mathematical expressions, one can compute with Quick Bayes estimators, and remarkably, the results of these calculations also encode arbitrary confidence intervals for the mathematical results. The Quick Bayes approach therefore allows analysts to compute with confidence, both figuratively and literally, because the performance properties of the Quick Bayes estimators can be projected to comparable performance properties for calculation results. This feature may make this approach useful in engineering because it offers a guarantee of statistical performance through repeated use.

Quick Bayes estimates can be computed in a variety of ways directly from random sample data which may be precise or imprecise. Estimators have been derived for both parametric problems where the distribution family is known to be normal, exponential, binomial, etc., and also for nonparametric problems in which the shape of the underlying distribution is unknown.

7. Conclusions

Modeling of epistemic uncertainty is, at first glance, an area of great diversity. Analysis, however, reveals consistency and compatibility between different concepts. Essentially, two pathways can be followed in modeling: using subjective probabilities and using set-theoretical descriptors. Subjective probabilities are usually employed in the context of the Bayesian approach, whereas set-theoretical descriptors lead to imprecise probabilities. The diversity of imprecise probabilities collapses into the overarching approaches of probability boxes and fuzzy probabilities. Imprecise probabilities and subjective probabilities show complementary features. The results from both approaches stand in relation to one another and are not necessarily contradictory. Whilst adding detail within the subjective probabilities approach affects the results from “the inside out” to the epistemic uncertainty, adding information in the imprecise probabilities approach affects the results from “the outside in” to the epistemic uncertainty. This may be regarded as a duality feature. Combinations of both approaches are also meaningful. In any case, the choice among the approaches should be driven by both the nature of the available information and the purpose of the analysis. To apply both approaches in parallel can reveal important additional information.

References

Au, S.-K. 2012. Fast Bayesian ambient modal identification in the frequency domain, Part I: posterior most probable value. *Mechanical Systems and Signal Processing*, 26: 60–75.

Au, S.-K. 2012. Fast Bayesian ambient modal identification in the frequency domain, Part II: posterior uncertainty. *Mechanical Systems and Signal Processing*, 26: 76–90.

Augustin, T., Coolen, F.P.A., de Cooman, G. and Troffaes, M.C.M. (Eds). 2014. *Introduction to Imprecise Probabilities*, Wiley.

Beer, M. 2009. Fuzzy probability theory. In *Encyclopedia of Complexity and Systems Science*, Vol. 6, Springer, New York, pp. 4047–4059.

Beer, M., Ferson, S. and Kreinovich, V. 2013. Imprecise probabilities in engineering analyses. *Mechanical Systems and Signal Processing*, 37(1–2): 4–29.

Beer, M., DiazDelaO, F.A., Patelli, E. and Au, S.-K. 2014. Conceptual comparison of Bayesian approaches and imprecise probabilities. In *Computational Technology Reviews* 9. Edited by Topping, B.H.V. and Ivanyi, P. Saxe-Coburg Publications, pp. 1–29.

Berger, J.O. 1994. An overview of robust Bayesian analysis (with discussion). *Test*, 3: 5–124.

Buckley, J.J., 2005. Fuzzy probabilities – new approach and applications. In *Studies in Fuzziness and Soft Computing*, Vol. 115, Springer, Berlin, Heidelberg.

Ellingwood, B.R. and Kinali, K. 2009. Quantifying and communicating uncertainty in seismic risk assessment. *Structural Safety*, 31: 179–187.

Ferson, S. and Hajagos, J.G. 2004. Arithmetic with uncertain numbers: rigorous and (often) best possible answers. *Reliability Engineering and System Safety*, 85(1–3): 135–152.

Ferson, S., Kreinovich, V., Ginzburg, L., Myers, D.S. and Sentz, K. 2003. Constructing Probability Boxes and Dempster–Shafer Structures. *Technical Report SAND2002-4015*. Sandia National Laboratories, Albuquerque, NM.

Ferson, S., O'Rawe, J. and Balch, M. 2014. Computing with Confidence: Imprecise Posteriors and Predictive Distributions. In *Vulnerability, Uncertainty and Risk – Quantification, Mitigation and Management*. Edited by Beer, M., Au, S.-K. and Hall, J.W. ASCE Council on Disaster Risk Management, Monograph No 9, CD-ROM, 895–904.

Klir, G.J. 2006. *Uncertainty and Information: Foundations of Generalized Information Theory*, Wiley-Interscience, Hoboken.

Moens, D. and Vandepitte, D. 2005. A survey of non-probabilistic uncertainty treatment in finite element analysis. *Computer Methods in Applied Mechanics and Engineering*, 194(1): 1527–1555.

Möller, B. and Beer, M. 2008. Engineering computation under uncertainty – capabilities of non-traditional models. *Computers and Structures*, 86(10): 1024–1041.

Mottershead, J.E., Khodaparast, H., Dwight, R.P. and Badcock, K.J. 2011. Stochastic Model Updating: Perturbation, Interval Method and Bayesian Inference. In *Springer Proceedings in Physics*, Vol. 139, Vibration Problems ICOVP 2011, Springer.

Patelli, E., Alvarez, D., Broggi, M. and De Angelis, M. 2015. Uncertainty Management in Multidisciplinary Design of Critical Safety Systems. *AIAA Journal of Aerospace Information Systems*, 12(1): 140–169.

Stein, M, Beer, M. and Kreinovich, V. 2013. Bayesian approach for inconsistent information. *Information Sciences*, 245: 96–111, 2013.

Walley, P. 1991. *Statistical Reasoning with Imprecise Probabilities*, Chapman & Hall, London.