How to Describe Measurement Uncertainty and Uncertainty of Expert Estimates?

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***Abstract*—Measurement and expert estimates are never absolutely accurate. Thus, when we know the result $M(u)$ of measurement or expert estimate, the actual value $A(u)$ of the corresponding quantity may be somewhat different from $M(u)$. In practical applications, it is desirable to know how different it can be, i.e., what are the bounds $f(M(u)) \leq A(u) \leq g(M(u))$. Ideally, we would like to know the tightest bounds, i.e., the largest possible values $f(x)$ and the smallest possible values $g(x)$. In this paper, we analyze for which (partially ordered) sets of values such tightest bounds always exist: it turns out that they always exist only for complete lattices.**

I. Formulation of the Problem

How can we describe measurement uncertainty: formulation of the problem. We want to know the actual values of different quantities. To get these values, we perform measurements.

Measurements are never absolutely accurate, there is always measurement uncertainty, in the sense that the actual value $A(u)$ of the corresponding physical quantity is, in general, different from the measurement result $M(u)$; see, e.g., [8].

This uncertainty means that the actual value $A(u)$ can be somewhat different from the measurement result $M(u)$. It is therefore desirable to describe what are the possible values of $A(u)$. This will be a perfect way to describe uncertainty: for each measurement result $M(u)$, we describe the set of all possible values of $A(u)$.

How can we attain this description?

Important remark: in practice, we do not know the actual values. Ideally, for different situations u, we should compare the measurement result $M(u)$ with the actual value $A(u)$. The problem is that we do not know the actual value – if we knew the actual value, we would not need to perform any measurements. So how do practitioners actually gauge the accuracy of measuring instruments?

A usual approach (see, e.g., [8]) is to compare the measurement result $M(u)$ with the result $S(u)$ of measuring the same quantity by using a much more accurate ("standard") measuring instrument. If the standard measuring instrument is indeed much more accurate than the one whose accuracy we are gauging, then, for the purpose of this gauging, we can:

- assume that $S(u) = A(u)$, and
- compare the results $M(u)$ and $S(u)$ of measuring the same quantity by two different measuring instruments.

In general, all we have is measurement results, so all we can do to gauge accuracy is to compare two measurement results. So, from the practical viewpoint, the above problem can be reformulated as follows:

- we know the measurement result $M(u)$ corresponding to some situation u,
- we would like to describe the set of possible values $S(u)$ that we would have obtained if we apply a standard measuring instrument to these same situation u.

Let us first list typical situations. Before we consider the general case, let us first describe several typical situations.

Case of absolute measurement error. In some cases, we know the upper bound Δ on the absolute value of the measurement error $M(u) - A(u)$, i.e., we know that

$$|M(u) - A(u)| \leq \Delta.$$

In this case, once we know the measurement result $M(u)$, we can conclude that the (unknown) actual value $A(u)$ satisfies the inequality

$$M(u) - \Delta \leq A(u) \leq M(u) + \Delta.$$

In other words, we conclude that $A(u)$ belongs to the *interval* $[M(u) - \Delta, M(u) + \Delta]$; see, e.g., [2], [4], [5].

In more general terms, we can describe the corresponding bounds as

$$f(M(u)) \leq A(u) \leq g(M(u)),$$

where

$$f(x) \stackrel{\text{def}}{=} x - \Delta \text{ and } g(x) \stackrel{\text{def}}{=} x + \Delta.$$

Case of relative measurement error. In some other cases, we know the upper bound δ on the *relative* measurement error:

$$\frac{|M(u) - A(u)|}{|A(u)|} \leq \delta.$$

In this case, for positive values,

$$(1 - \delta) \cdot A(u) \leq M(u) \leq (1 + \delta) \cdot A(u).$$

Thus, once we know the measurement result $M(u)$, we can conclude that the actual (unknown) value $A(u)$ of the corresponding physical quantity satisfies the inequality

$$\frac{M(u)}{1+\delta} \le A(u) \le \frac{M(u)}{1-\delta}.$$

In other words, we have

$$f(M(u)) \le A(u) \le g(M(u))$$

for

$$f(x) \stackrel{\text{def}}{=} \frac{x}{1+\delta} \text{ and } g(x) \stackrel{\text{def}}{=} \frac{x}{1-\delta}.$$

In some cases, we have both types of measurement errors. In some cases, we have both:

- *additive* measurement errors, i.e., errors whose absolute value does not exceed Δ, and
- *multiplicative* measurement errors, i.e., errors whose relative value does not exceed δ and thus, whose absolute value does not exceed $\delta \cdot |A(u)|$.

In this case, for positive values, we get

$$A(u) - \Delta - \delta \cdot A(u) \le M(u) \le A(u) + \Delta + \delta \cdot A(u).$$

The left inequality can be reformulated as

$$A(u) \cdot (1-\delta) - \Delta \le M(u),$$

hence

$$A(u) \cdot (1-\delta) \le M(u) + \Delta$$

and thus,

$$A(u) \le \frac{M(u)+\Delta}{1-\delta}.$$

Similarly, the right inequality can be reformulated as

$$M(u) \le A(u) \cdot (1+\delta) + \Delta,$$

hence

$$A(u) \cdot (1+\delta) \ge M(u) - \Delta$$

and thus,

$$A(u) \ge \frac{M(u)-\Delta}{1+\delta}.$$

In this case, we have

$$f(M(u)) \le A(u) \le g(M(u)),$$

where

$$f(x) \stackrel{\text{def}}{=} \frac{x-\Delta}{1+\delta} \text{ and } f(x) \stackrel{\text{def}}{=} \frac{x+\Delta}{1-\delta}.$$

Towards a general case. The above formulas assume that parameters Δ and δ describing measurement accuracy are the same for the whole range. In reality, measuring instruments have different accuracies in different ranges. Hence, the resulting functions $f(x)$ and $g(x)$ are non-linear.

It should be mentioned that all the above functions $f(x)$ and $g(x)$ are monotonic, and this is usually true for all measuring instruments: when the measurement result is larger, this usually means that the bounds on possible values of the actual quantity also increase (or at least do not decrease).

To describe the accuracy of a general measuring instrument, it is therefore reasonable to use:

- the largest of the monotonic functions $f(x)$ for which $f(M(u)) \le A(u)$ and
- the smallest of the monotonic functions $g(x)$ for which $A(u) \le g(M(u))$.

Similarly, to describe the relative accuracy of a measuring instrument $M(u)$ in comparison to a standard measuring instrument $S(u)$, it is reasonable to use:

- the largest of the monotonic functions $f(x)$ for which $f(M(u)) \le S(u)$ and
- the smallest of the monotonic functions $g(x)$ for which $S(u) \le g(M(u))$.

From measurements to expert estimates. While measurement are very important, a large part of our knowledge comes from *expert estimates*. Expert estimates are extremely important in areas such a medicine.

In contrast to measurements that always result in numbers, expert estimates often can also result in "values" from a partially ordered set. For example, when a medical doctor is asked how probable is a certain diagnosis, the doctor may provide an approximate probability, or an interval of possible probabilities, or a natural-language term like "somewhat probable" or "very probable".

Such possibilities are described, e.g., in different generalizations and extensions of the traditional $[0,1]$-based fuzzy logic; see, e.g., [3], [6], [10]; see also [1]. What is in common for all such extensions is that on the corresponding set of value L, there is always an *order* (sometimes partial), so that $\ell < \ell'$ means that ℓ' represents a stronger expert's degree of confidence.

Need to describe uncertainty of expert estimates. Some experts are very good, in the sense that based on their estimates, we make very effective decisions. These experts can be viewed as analogs of standard measuring instruments.

Other experts may be less accurate. It is therefore desirable to gauge the uncertainty of such experts in relation to the "standard" (very good) ones. If a regular expert provides an estimate $M(u)$ for a situation u, then, to make a good decision based on this estimate, we would like to know what would the perfect expert conclude in this case, i.e., what are the bounds on the perfect expert's estimates $S(u)$? In general, we may have several functions $f(x)$ and $g(x)$ for which

$$f(M(u)) \le S(u) \le g(M(u)).$$

It is desirable to find:

- the largest of the monotonic functions $f(x)$ for which $f(M(u)) \le S(u)$ and
- the smallest of the monotonic functions $g(x)$ for which $S(u) \le g(M(u))$.

What is known and what we do in this paper. For the case when the set L is an interval – e.g., the interval $[0,1]$ – the existence of the largest $f(x)$ and smallest $g(x)$ was proven in [7] (see also [9]).

In this paper, we analyze for which partially ordered sets such largest $f(x)$ and smallest $g(x)$ exist. It turns out that they exist for complete lattices – and, in general, do not exist for more general partially ordered sets. To be more precise,

- the largest $f(x)$ exists for complete lower semi-lattices (precise definitions are given below), while
- the smallest $g(x)$ exists for complete upper semi-lattices.

II. Main Result: For Lattices, It Is Possible to Describe Uncertainty in Terms of the Bounding Functions $f(x)$ and $g(x)$

Definition 1. *Let L be a (partially) ordered set, and let U be any set. We say that a function $F : U \to L$ is* smaller *that a function $G : U \to L$ is $F(u) \le G(u)$ for all $u \in U$. We will denote this by $F \le G$.*

Definition 2. *We say that a function $L \to L$ is* monotonic *if $x \le y$ implies $f(x) \le f(y)$.*

Notation 1. *For every ordered set L, by M_L, we denote the set of all monotonic functions $f : L \to L$.*

Definition 3. *Let $f \in M_L$. We say that a function $F : U \to L$ is f-*smaller *than a function $G : U \to L$ if $f(F(u)) \le G(u)$ for all $u \in U$. We will denote this by $F \le_f G$.*

Notation 2. *By $\mathcal{F}(F,G)$ we will denote the set of all functions $f \in M_L$ for which $F \le_f G$.*

Definition 4. *Let L be an ordered set, and let $S \subseteq L$ be its subset.*

- *We say that an element x is a* lower bound *for the set S if $x \le s$ for all $s \in S$.*
- *An ordered set is called a* complete lower semi-lattice *if for every set S, among all its lower bounds, there exists the largest one. This largest lower bound is denoted by $\bigwedge S$.*
- *We say that an element x is an* upper bound *for the set S if $s \le x$ for all $s \in S$.*
- *An ordered set is called a* complete upper semi-lattice *if for every set S, among all its upper bounds, there exists the smallest one. This smallest upper bound is denoted by $\bigvee S$.*
- *An ordered set L is called a* complete lattice *if it is both a complete lower semi-lattice and a complete upper semi-lattice.*

Proposition 1. *If L is a complete lower semi-lattice, then for every two functions $F, G : U \to L$, the set $\mathcal{F}(F,G)$ has the largest element $f_{F,G}$ for which*

$$\mathcal{F}(F,G) = \{f \in M_L : f \le f_{F,G}\}.$$

Proof. We will prove that the function

$$f_{F,G}(x) \stackrel{\text{def}}{=} \bigwedge \{G(u) : x \le F(u)\}$$

is the desired function. In other words, we will prove:

- that the function $f_{F,G}$ belongs to the class $\mathcal{F}(F,G)$ and
- that the function $f_{F,G}$ is the largest function in this class.

Let us first prove that $f_{F,G} \in \mathcal{F}(F,G)$, i.e., that for every u, we have $f_{F,G}(F(u)) \le G(u)$. Indeed, for $x = F(u)$, we have $x \le F(u)$, and thus, the element $G(u)$ belongs to the set $\{G(u) : x \le F(u)\}$. Thus, this element $G(u)$ is larger than or equal to the largest lower bound

$$f_{F,G}(x) = \bigwedge \{G(u) : x \le F(u)\}$$

for this set, i.e., indeed

$$f_{F,G}(F(u)) = f_{F,G}(x) \le G(u).$$

Let us now prove that the function $f_{F,G}$ is the largest function in the class $\mathcal{F}(F,G)$, i.e., that if $f \le \mathcal{F}(F,G)$, then $f \le f_{F,G}$. Indeed, let $f \in \mathcal{F}(F,G)$. By definition of this class, this means that f is monotonic and $f(F(u)) \le G(u)$ for all u. Let us pick some $x \in L$ and show that $f(x) \le f_{F,G}(x)$. Indeed, for every value $u \in U$ for which $x \le F(u)$, we have, due to monotonicity, $f(x) \le f(F(u))$. Since $f(F(u)) \le G(u)$, we this conclude that $f(x) \le G(u)$. So, the value $f(x)$ is smaller than or equal to all elements of the set $\{G(u) : x \le F(u)\}$, i.e., $f(x)$ is a lower bound for this set. Every lower bound is smaller than or equal to the largest lower bound

$$f_{F,G}(x) = \bigwedge \{G(u) : x \le F(u)\},$$

so indeed $f(x) \le f_{F,G}(x)$.

Let us now prove that $\mathcal{F}(F,G) = \{f \in M_L : f \le f_{F,G}\}$. We have shown that every function $f \in \mathcal{F}(F,G)$ is $\le f_{F,G}$, i.e., that

$$\mathcal{F}(F,G) \subseteq \{f \in M_L : f \le f_{F,G}\}.$$

Vice versa, if $f \le f_{F,G}$, then for every u, from $f_{F,G}(F(u)) \le G(u)$ and $f(F(u)) \le f_{F,G}(F(u))$, we conclude that $f(F(u)) \le G(u)$, i.e., that indeed $f \in \mathcal{F}(F,G)$.

The proposition is proven.

Discussion. A similar result can be obtained for the upper bounds.

Definition 5. *Let $f \in M_L$. We say that a function $G : U \to L$ is g-*larger *than a function $F : U \to L$ if $F(u) \le g(G(u))$ for all $u \in U$. We will denote this by $G \ge_g F$.*

Notation 3. *By $\mathcal{G}(F,G)$ we will denote the set of all functions $g \in M_L$ for which $G \ge_g F$.*

Proposition 2. *If L is a complete upper semi-lattice, then for every two functions $F, G : U \to L$, the set $\mathcal{G}(F,G)$ has the smallest element $g_{F,G}$ for which*

$$\mathcal{G}(F,G) = \{g \in M_L : g \ge g_{F,G}\}.$$

Proof is similar to the proof of Proposition 1.

III. The Main Result Cannot Be Extended Beyond Complete Lower Semi-Lattices

Let us prove that this result cannot be extended beyond complete semi-lattices.

Proposition 3. *Let L be an ordered set for which, for every two functions $F, G : U \to L$, the set $\mathcal{F}(F, G)$ has the largest element. Then the set L is a complete lower semi-lattice.*

Proof. Let us assume that the ordered set L has the above property. Let us prove that L is a complete lower semi-lattice. Indeed, let $S \subseteq L$ be any subset of L. Let us take $U = S$, and take $G(u) = u$ for all $u \in S$. Let us also pick any element $x_0 \in L$ and take $F(u) = x_0$ for all $u \in S$. Because of our assumption, the set $\mathcal{F}(F, G)$ of all the functions for which $f(F(u)) \le G(u)$ for all u has the largest element $f_{F,G}$.

Because of our choice of the functions $F(u)$ and $G(u)$, the inequality $f(F(u)) \le G(u)$ simply means that $f(x_0) \le u$ for all $u \in S$, i.e., that $f(x_0)$ is the lower bound for the set S. The fact that there is the largest of such functions $f \in \mathcal{F}(F, G)$ means that there is the largest of the lower bounds – which is exactly the definition of the complete lower semi-lattice. The proposition is proven.

Proposition 4. *Let L be an ordered set for which, for every two functions $F, G : U \to L$, the set $\mathcal{G}(F, G)$ has the smallest element. Then the set L is a complete upper semi-lattice.*

Proof is similar to the proof of Proposition 3.

IV. Auxiliary Results: What If There Is No Bias?

Comment about bias. In some practical situations, measuring instrument has a *bias* (shift): a clock can be regularly 2 minutes behind, a thermometer can regularly show temperatures which are 3 degrees higher, etc. Bias means that we get the measurement result $M(u)$, then this value *cannot* be equal to the actual value of the measured quantity: there is always a non-zero shift $A(u) - M(u)$.

Bias can easily be eliminated by re-calibrating the measuring instrument: for example, if I move to a different time zone, I can simply add or subtract the corresponding time difference and get the exact local time.

It is therefore reasonable to assume that the bias has already been eliminated, and that, thus, $A(u) = M(u)$ is one of the possible actual values. For this value $A(u) = M(u)$, our inequality

$$f(M(u)) \le A(u) \le g(M(u))$$

implies that

$$f(x) \le x \le g(x).$$

So, it makes sense to only consider functions $f(x)$ and $g(x)$ for which $f(x) \le x$ and $x \le g(x)$. It turns out that similar results hold when we thus restrict the functions $f(x)$ and $g(x)$.

Notation 4. *For every ordered set L, by Ω_L, we denote the set of all monotonic functions $f : L \to L$ for which $f(x) \le x$ for all $x \in L$.*

Notation 5. *By $\mathcal{F}_u(F, G)$ we will denote the set of all functions $f \in \Omega_L$ for which $F \le_f G$.*

Proposition 5. *If L is a complete lower semi-lattice, then for every two functions $F, G : U \to L$, the set $\mathcal{F}_u(F, G)$ has the largest element $f_{F,G}$ for which*

$$\mathcal{F}_u(F, G) = \{f \in \Omega_L : f \le f_{F,G}\}.$$

Proof. We will prove that the function

$$f_{F,G}(x) \stackrel{\text{def}}{=} \bigwedge \{G(u) \wedge x : x \le F(u)\}$$

is the desired function. In other words, we will prove:

- that the function $f_{F,G}$ belongs to the class $\mathcal{F}_u(F, G)$ and
- that the function $f_{F,G}$ is the largest function in this class.

Let us first prove that $f_{F,G} \in \mathcal{F}_u(F, G)$, i.e., that for every u, we have $f_{F,G}(F(u)) \le G(u)$. Indeed, for $x = F(u)$, we have $x \le F(u)$, and thus, the element $G(u)$ belongs to the set $\{G(u) : x \le F(u)\}$. Thus, this element $G(u)$ is larger than or equal to the largest lower bound

$$f_{F,G}(x) = \bigwedge \{G(u) : x \le F(u)\}$$

for this set, i.e., indeed

$$f_{F,G}(F(u)) = f_{F,G}(x) \le G(u).$$

Since $G(u) \wedge x \le x$, we conclude that $f_{F,G}(x) \le x$. Thus, indeed, $f_{F,G} \in \Omega_L$.

Let us now prove that the function $f_{F,G}$ is the largest function in the class $\mathcal{F}_u(F, G)$, i.e., that if $f \le \mathcal{F}_u(F, G)$, then $f \le f_{F,G}$. Indeed, let $f \in \mathcal{F}_u(F, G)$. By definition of this class, this means that f is monotonic, $f(x) \le x$ or all x, and $f(F(u)) \le G(u)$ for all u. Let us pick some $x \in L$ and show that $f(x) \le f_{F,G}(x)$. Indeed, for every value $u \in U$ for which $x \le F(u)$, we have, due to monotonicity, $f(x) \le f(F(u))$. Since $f(F(u)) \le G(u)$, we this conclude that $f(x) \le G(u)$. So, the value $f(x)$ is smaller than or equal to all elements of the set $\{G(u) : x \le F(u)\}$, i.e., $f(x)$ is a lower bound for this set. Moreover, as $f(x) \le x$, we have

$$f(x) \le \bigwedge \{G(u) \wedge x : x \le F(u)\} = f_{F,G}(x),$$

so indeed $f(x) \le f_{F,G}(x)$.

Let us now prove that

$$\mathcal{F}_u(F, G) = \{f \in \Omega_L : f \le f_{F,G}\}.$$

We have shown that every function $f \in \mathcal{F}_u(F, G)$ is $\le f_{F,G}$, i.e., that

$$\mathcal{F}(F, G) \subseteq \{f \in \Omega_L : f \le f_{F,G}\}.$$

Vice versa, if $f \le f_{F,G}$, then for every u, from $f_{F,G}(F(u)) \le G(u)$ and $f(F(u)) \le f_{F,G}(F(u))$, we conclude that $f(F(u)) \le G(u)$, i.e., that indeed $f \in \mathcal{F}_u(F, G)$.

The proposition is proven.

Notation 6. *For every ordered set L, by Θ_L, we denote the set of all monotonic functions $g : L \to L$ for which $x \le g(x)$ for all $x \in L$.*

Notation 7. *By $\mathcal{G}_u(F,G)$ we will denote the set of all functions $g \in \Theta_L$ for which $G \ge_g F$.*

Proposition 6. *If L is a complete upper semi-lattice, then for every two functions $F, G : U \to L$, the set $\mathcal{G}_u(F,G)$ has the smallest element $g_{F,G}$ for which*

$$\mathcal{G}_u(F,G) = \{g \in \Theta_L : g \ge g_{F,G}\}.$$

Proof is similar to the proof of Proposition 5.

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