How to Transform Partial Order Between Degrees into Numerical Values

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***Abstract*—Fuzzy techniques are a successful way to handle expert knowledge, enabling us to capture different degrees of experts' certainty in their statements. To use fuzzy techniques, we need to describe experts' degrees of certainty in numerical terms. Some experts can provide such numbers, but others can only describe their degrees by using natural-language words like "very", "somewhat", "to some extent", etc. In general, all we know about these word-valued degrees is that there is a natural partial order between these degrees: e.g., "very small" is clearly smaller than "somewhat small". In this paper, we propose a natural way to transform such a partial order between degrees into numerical values.**

I. Formulation of the Problem

Why fuzzy logic: a brief reminder. In many practical situations, there are experts who are skilled in performing the corresponding task:

- skilled machine operators successfully operate machinery,
- skilled medical doctors successfully cure patients, etc.

So, if we design an automated system that would replace these experts – or at least help less skilled operators and doctors make proper decisions – it is important to incorporate the knowledge of the experts into this system.

Some of this expert knowledge can be described in precise ("crisp") form. Such knowledge is relative easy to describe in precise computer-understandable terms. However, a significant part of human knowledge is described in imprecise ("fuzzy") terms like "small", "fast", etc. One of the main objectives of fuzzy logic is to translate this knowledge into precise machine-understandable form. For this purpose, for each imprecise (fuzzy) natural-language property like "small", Zadeh proposed to describe, for each possible value of the corresponding quantity, the degree to which this value satisfies the selected property (e.g., to which extent the given value x is small); see, e.g., [5], [13], [14].

Intuitively, we often describe such degrees by using words from natural language, such as "very small", "somewhat small", etc. However, computers are not very good in precessing natural-language terms, they are much more efficient in processing numbers. As a result, the corresponding fuzzy techniques require that we first translate the corresponding degrees into numbers from the interval $[0, 1]$.

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Sometimes, the corresponding degrees are difficult to elicit. Some experts can easily describe their degrees in terms of numbers, but other experts are more comfortable describing degrees in natural-language terms.

In this case, we need to translate the resulting terms into numbers from the interval $[0, 1]$.

What information we can use for this translation. For some pairs of degrees, we know which degree corresponds to a larger confidence. For example, it is clear that "very small" is smaller than "somewhat small".

It is reasonable to assume that these expert comparisons are transitive and cycle-free. Thus, we usually have a natural (partial) *order* relation between different degrees.

This order is not necessarily total (linear): we may have two degrees with no relation between them, e.g., "reasonably small" and "to some extent small". Thus, in general, the corresponding relation is a *partial order*. We therefore arrive at the following problem.

Formulation of the problem. We have a finite partially ordered set. We would like to assign numbers from the interval $[0, 1]$ to different elements from this set – in such a way that if $a < b$ then the number assigned to a should be smaller than the number assigned to b.

Of course, there are many such possible assignments. Our goal is to select the assignment which is, in some sense, the most reasonable. This is what we will do in this paper.

II. Main Idea

What we want: towards a precise formulation. Let us number the elements of the original finite partially ordered set by numbers 1, 2, ..., k. For simplicity, let us identify each element of the given finite set with the corresponding number. After this identification, we get the set $\{1, 2, \ldots, k\}$ with some partial order $\prec$. (This order is, in general, different from the natural order $<$).

The desired mapping means that we assign, to each of the numbers i from 1 to k, a real number $x_i \in [0, 1]$. In other words, the mapping means that we produce a tuple $x = (x_1, \ldots, x_k)$ of real numbers from the interval $[0, 1]$.

The only restriction on this tuple is that if $i \prec j$, then $x_i < x_j$. Let us denote the set of all the tuples x that satisfy this restriction by $S_\prec$.

Our main idea. Out of many possible tuples from the set $S_\prec$, we would like to select one $s = (s_1, \ldots, s_k)$. Which one should we select?

Selecting a tuple means that we need to select, for each i, the corresponding value s_i. The ideally-matching tuple x has, in general, a different value $x_i \neq s_i$. It usually makes sense to describe the inaccuracy ("loss") of this selection by the square $(s_i - x_i)^2$ of the difference between the selected and the ideal values.

We do not know what is the ideal value x_i, we only know that this ideal value is the i-th component of some tuple $x \in S_\prec$. We have no reason to believe that some tuples are more probable than the others. As a result, it makes sense to consider them all equally probable. So, if we select the tuple s, then the expected loss is proportional to

$$\int_{S_\prec} (x_i - s_i)^2 \, dx. \tag{1}$$

It is therefore reasonable to select a value s_i for which this loss is the smallest possible, i.e., for which

$$\int_{S_\prec} (x_i - s_i)^2 \, dx \to \min_s. \tag{2}$$

III. From the Idea to an Algorithm

Need for an algorithm. Our objective is to come up with numbers describing expert degrees. From this viewpoint, the formulation (2) is somewhat over-complicated. What we need is a simple algorithm that would transform the partial order on the set of degrees into numerical values.

Let us show how to come up with such an algorithm.

First step: using calculus. As a first step towards the desired algorithm, let us use the standard calculus-based criterion for optimality: namely, let us differentiate the objective function (1) with respect to s_i and equate the resulting derivative to 0. As a result, we get the expression

$$\int_{S_\prec} (s_i - x_i) \, dx = 0,$$

hence

$$s_i \cdot \left(\int_{S_\prec} dx \right) = \int_{S_\prec} x_i \, dx,$$

and

$$s_i = \frac{N}{D}, \tag{3}$$

where

$$N \stackrel{\text{def}}{=} \int_{S_\prec} x_i \, dx \tag{4}$$

and

$$D \stackrel{\text{def}}{=} \int_{S_\prec} dx. \tag{5}$$

From this viewpoint, to compute s_i, it is sufficient to compute the corresponding integrals (4) and (5).

How to compute the corresponding integrals: idea and algorithm. Since $\prec$ is a partial order, in the set $S_\prec$, in general, we may have tuples $(x_1, \ldots, x_k)$ with different orderings between elements x_i. For example, if we know only that $1 \prec 2$ and $1 \prec 3$, but we do not know of any relation between 2 and 3, then we can have two possible orderings: $1 \prec 2 \prec 3$ and $1 \prec 3 \prec 2$.

(In principle, we can also have equalities between the values x_i, but the areas in which two values are equal have 0 volume and thus, can be ignored when computing the integrals.)

In general, there are finitely many ($k!$) possible linear orders between the values $x_1, \ldots, x_k$, some of them may be consistent with $\prec$. Let us denote the set of all the tuples with an order ℓ by T_ℓ. Then, each set $P_\prec$ is the union of the sets T_ℓ for all linear orders ℓ extending $\prec$, i.e.,

$$S_\prec = \bigcup_{\ell : \ell \supseteq \prec} T_\ell,$$

where $\ell \supseteq \prec$ means that the linear order ℓ extends the partial order $\prec$.

Thus, each of the integrals N and D over the set $S_\prec$ can be represented as the sum of integrals over the sets T_ℓ:

$$D = \sum_{\ell : \ell \supseteq \prec} D_\ell, \tag{6}$$

where

$$D_\ell \stackrel{\text{def}}{=} \int_{T_\ell} x_i \, dx, \tag{7}$$

and

$$N = \sum_{\ell : \ell \supseteq \prec} N_\ell, \tag{8}$$

where

$$N_\ell \stackrel{\text{def}}{=} \int_{T_\ell} dx. \tag{9}$$

Thus, to find s_i, it is sufficient to be able to compute the corresponding integrals (7) and (9).

Each of these integrals can be computed by integrating variable-by-variable. For each of the variables x_j, we integrate a polynomial with rational coefficients, and the ranges are between some values x_m and x_n, so the integral is still a polynomial. After all integrations, we get a rational number.

By adding the resulting rational numbers D_ℓ and N_ℓ, we get D and N and thus, by dividing them, we get the desired degree s_i.

Actually, the value D_ℓ can be computed even faster: the integral D_ℓ is simply the volume of the set S_ℓ. The unit cube $[0, 1]^k$ of volume 1 is divided into $k!$ such parts of equal volume, so $D_\ell = \dfrac{1}{k!}$.

Comment. Please note that for each total order ℓ, the denominator D_ℓ does not depend on which of the k values s_i we compute and is, therefore, the same for all i.

Examples follow. The above description may be somewhat complicated and not perfectly clear.

In the following sections, we provide several step-by-step examples that will hopefully make the above algorithm much clearer.

IV. Example 1: A 1-Element Set

Description of the situation. Let us start with the simplest possible case, when we have a single degree, i.e., in the mathematical terms, when the partially ordered set consists of a single element 1.

In this case, what we want is to find the single degree s_1.

Applying our algorithm. In this case, there is no order, so there are no restrictions on the values x_1. Thus, we have only one set T_ℓ which simply coincides with the interval $[0,1]$. For this set,

$$N = \int_0^1 x_1\,dx_1 = \frac{1}{2}\cdot x_1^2\Big|_0^1 = \frac{1}{2}$$

and

$$D = \int_0^1 dx_1 = x_1|_0^1 = 1,$$

thus,

$$s_1 = \frac{N}{D} = \frac{1}{2}.$$

Result. In a situation when we know nothing about the degree, our idea leads to selecting $s_1 = 0.5$.

Discussion. This selection makes sense: we have no reason to believe whether $x < s_1$ or $x > s_1$, and indeed, this selection divides the whole interval $[0,1]$ into two equal parts: values below s_1 and values above s_1.

V. Example 2: A 2-Element Set With No Order

If we have two unrelated degrees 1 and 2, then we can repeat the same argument for each of these sets and conclude that

$$s_1 = s_2 = \frac{1}{2}.$$

VI. Example 3: An Ordered 2-Element Set

Description of the situation. Let us now consider the situation in which we have two ordered degrees. Without losing generality, we can assume that $1 \prec 2$. In this case, we need to compute two values $s_1 < s_2$ that correspond to these two degrees.

Applying our algorithm to compute s_1. In this situation, we have only one order ℓ: $1 \prec 2$. So, T_ℓ is the set of all the pairs (x_1, x_2) for which $x_1 < x_2$. Thus, x_2 can take any value from the interval $[0,1]$, and, once x_2 is fixed, x_1 can take any value from 0 to x_2:

$$N_\ell = \int_{0\le x_1 < x_\le 1} x_1\,dx = \int_0^1 dx_2 \int_0^{x_2} x_1\,dx_1. \qquad (10)$$

The inner integral has the form

$$\int_0^{x_2} x_1\,dx_1 = \frac{1}{2}\cdot x_1^2\Big|_0^{x_2} = \frac{1}{2}\cdot x_2^2,$$

thus,

$$N_\ell = \int_0^1 dx_2 \int_0^{x_2} x_1\,dx_1 = \int_0^1 dx_2 \cdot \frac{1}{2}\cdot x_2^2 = \frac{1}{6}\cdot x_2^3\Big|_0^1 = \frac{1}{6}.$$

Similarly,

$$D_\ell = \int_{0\le x_1 < x_2 \le 1} dx = \int_0^1 dx_2 \int_0^{x_2} dx_1. \qquad (11)$$

The inner integral has the form

$$\int_0^{x_2} dx_1 = x_1|_0^{x_2} = x_2,$$

thus,

$$D_\ell = \int_0^1 dx_2 \int_0^{x_2} dx_1 = \int_0^1 dx_2 \cdot x_2 = \frac{1}{2}\cdot x_2^2\Big|_0^1 = \frac{1}{2}.$$

Therefore, we get

$$s_1 = \frac{N}{D} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}.$$

Applying our algorithm to compute s_2. Here,

$$N_\ell = \int_{0\le x_1 < x_2 \le 1} x_2\,dx = \int_0^1 x_2\cdot dx_2 \int_0^{x_2} dx_1. \qquad (12)$$

We already know that the inner integral has the form

$$\int_0^{x_2} dx_1 = x_2,$$

thus

$$N_\ell = \int_0^1 x_2\cdot dx_2 \int_0^{x_2} dx_1 = \int_0^1 x_2\cdot dx_2 \cdot x_2 =$$

$$\int_0^1 x_2^2\,dx_2 = \frac{1}{3}\cdot x_2^3\Big|_0^1 = \frac{1}{3},$$

and

$$s_2 = \frac{N}{D} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.$$

Result. In a situation when we have two ordered degrees $1 \prec 2$, we select $s_1 = \frac{1}{3}$ and $s_2 = \frac{2}{3}$.

Discussion.

- This selection also makes sense, since we divide the interval $[0,1]$ into 3 equal zones: below s_1, between s_1 and s_2, and above s_2.
- For the case of linearly ordered set, a similar idea was used in [6], [7], [8], [9] to explain a seemingly irrational character of human decision making under uncertainty – as described, e.g., in [4].

VII. Example 4: A 3-Element Set With No Order

We have considered all possible cases of 2-element sets, let us now consider 3-element ordered sets.

The first case is when we have three unrelated degrees 1, 2, and 3. In this case, we can repeat the same argument for each of these sets and conclude that

$$s_1 = s_2 = s_3 = \frac{1}{2}.$$

VIII. Example 5: A 3-Element Set With Two Elements Ordered and One Unrelated

In this case, for the unrelated element 1, we get $s_1 = \frac{1}{2}$, and for the two ordered elements $2 \prec 3$, we get $s_2 = \frac{1}{3}$ and $s_3 = \frac{2}{3}$.

IX. Example 6: A Totally (Linearly) Ordered 3-Element Set

Description of the situation. Let us now consider the situation in which we have three totally (linearly) ordered degrees. Without losing generality, we can assume that $1 \prec 2 \prec 3$. In this case, we need to compute three values $s_1 < s_2 < s_3$ that corresponding to these three degrees.

Applying our algorithm to compute s_1. In this situation, we have only one order ℓ: $1 \prec 2 \prec 3$. So, T_ℓ is the set of all the triples (x_1, x_2, x_3) for which $x_1 < x_2 < x_3$. Thus:

- the variable x_3 can take any value from the interval $[0, 1]$,
- once x_3 is fixed, the variable x_2 can take any value from 0 to x_3, and
- once x_3 and x_2 are fixed, x_1 can take any value from 0 to x_2.

Thus:

$$N_\ell = \int_{0 \le x_1 < x_2 < x_3 \le 1} x_1 \, dx =$$

$$\int_0^1 dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} x_1 \, dx_1. \quad (13)$$

We already know that the inner integral is equal to

$$\int_0^{x_2} x_1 \, dx_1 = \frac{1}{2} \cdot x_2^2,$$

thus,

$$N_\ell = \int_0^1 dx_3 \int_0^{x_3} dx_2 \cdot \frac{1}{2} \cdot x_2^2.$$

Now, the integral over x_2 has the form

$$\int_0^{x_3} dx_2 \cdot \frac{1}{2} \cdot x_2^2 = \frac{1}{6} \cdot x_2^3 \Big|_0^{x_3} = \frac{1}{6} \cdot x_3^3$$

and therefore,

$$N_\ell = \int_0^1 \frac{1}{6} \cdot x_3^3 = \frac{1}{24} \cdot x_3^3 \Big|_0^1 = \frac{1}{24}.$$

Here, $D_\ell = \frac{1}{3!} = \frac{1}{6}$. Thus,

$$s_1 = \frac{\frac{1}{24}}{\frac{1}{6}} = \frac{1}{4}.$$

Applying our algorithm to compute s_2. Here:

$$N_\ell = \int_{0 \le x_1 < x_2 < x_3 \le 1} x_2 \, dx =$$

$$\int_0^1 dx_3 \int_0^{x_3} x_2 \, dx_2 \int_0^{x_2} dx_1. \quad (14)$$

We already know that the inner integral is equal to

$$\int_0^{x_2} dx_1 = x_2,$$

thus,

$$N_\ell = \int_0^1 dx_3 \int_0^{x_3} x_2 \, dx_2 \cdot x_2.$$

Now, the integral over x_2 has the form

$$\int_0^{x_3} dx_2 \cdot x_2^2 = \frac{1}{3} \cdot x_2^3 \Big|_0^{x_3} = \frac{1}{3} \cdot x_3^3$$

and therefore,

$$N_\ell = \int_0^1 \frac{1}{3} \cdot x_3^3 = \frac{1}{12} \cdot x_3^3 \Big|_0^1 = \frac{1}{12}.$$

Thus,

$$s_2 = \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2}.$$

Applying our algorithm to compute s_3. Here:

$$N_\ell = \int_{0 \le x_1 < x_2 < x_3 \le 1} x_3 \, dx =$$

$$\int_0^1 x_3 \, dx_3 \int_0^{x_3} dx_2 \int_0^{x_2} dx_1. \quad (15)$$

We already know that the inner integral is equal to

$$\int_0^{x_2} dx_1 = x_2,$$

thus,

$$N_\ell = \int_0^1 x_3 \cdot dx_3 \int_0^{x_3} dx_2 \cdot x_2.$$

Now, the integral over x_2 has the form

$$\int_0^{x_3} x_2 \cdot dx_2 = \frac{1}{2} \cdot x_2^2 \Big|_0^{x_3} = \frac{1}{2} \cdot x_3^2$$

and therefore,

$$N_\ell = \int_0^1 \frac{1}{2} \cdot x_3^3 = \frac{1}{8} \cdot x_3^3 \Big|_0^1 = \frac{1}{8}.$$

Thus,

$$s-1=\frac{\frac{1}{8}}{\frac{1}{6}}=\frac{3}{4}.$$

Result. In a situation when we have three linearly degrees $1 \prec 2 \prec 3$, we select $s_1=\frac{1}{4}$, $s_2=\frac{1}{2}$, and $s_3=\frac{3}{4}$.

Discussion. This selection makes sense, since we divide the interval $[0,1]$ into 4 equal zones: below s_1, between s_1 and s_2, between $s-2$ and s_3, and above s_3.

X. Example 7: Partially Ordered 3-Element Set

Description of the situation. Let us now consider situations in which we have a partially (not linearly) ordered 3-element set in which no element is isolated. There are two possible options:

- an option in which $1 \prec 2$ and $1 \prec 3$, but 2 and 3 are not related, and
- an option in which $1 \prec 3$, $2 \prec 3$, but 1 and 2 are not related.

Let us first consider the first option. Let us start by considering the first option. In this option, we have two possible linear orders:

- the order ℓ' in which $1 \prec 2 \prec 3$ and
- the order ℓ'' in which $1 \prec 3 \prec 2$.

Thus, here, $D_\prec = D_{\ell'} + D_{\ell''} = 2 \cdot D_\ell$, and $N_\prec = N_{\ell'} + N_{\ell''}$. Thus, for each i, we have

$$s_i=\frac{N_\prec}{D_\prec}=\frac{N_{\ell'}+N_{\ell''}}{2D_\ell}=\frac{1}{2}\cdot\left(\frac{N_{\ell'}}{D_{\ell'}}+\frac{N_{\ell''}}{D_{\ell''}}\right)=\frac{s'_i+s''_i}{2},$$

where s'_i and s''_i are the values corresponding to the orders ℓ' and ℓ''.

We know, from the previous section, that

$$s'_1=\frac{1}{4},\quad s'_2=\frac{1}{2},\quad s'_3=\frac{3}{4};$$

$$s''_1=\frac{1}{4},\quad s''_2=\frac{3}{4},\quad s''_3=\frac{1}{2}.$$

Thus, we get

$$s_1=\frac{s'_1+s''_1}{2}=\frac{\frac{1}{4}+\frac{1}{4}}{2}=\frac{1}{4};$$

$$s_2=\frac{s'_2+s''_2}{2}=\frac{\frac{1}{2}+\frac{3}{4}}{2}=\frac{5}{8};$$

$$s_3=\frac{s'_3+s''_3}{2}=\frac{\frac{3}{4}+\frac{1}{2}}{2}=\frac{5}{8}.$$

Second option. In this case, we also have two possible linear orders:

- the order ℓ' in which $1 \prec 2 \prec 3$ and
- the order ℓ'' in which $2 \prec 1 \prec 3$.

Thus, we similarly get

$$s_1=\frac{s'_1+s''_1}{2}=\frac{\frac{1}{4}+\frac{1}{2}}{2}=\frac{3}{8};$$

$$s_2=\frac{s'_2+s''_2}{2}=\frac{\frac{1}{2}+\frac{1}{4}}{2}=\frac{3}{8};$$

$$s_3=\frac{s'_3+s''_3}{2}=\frac{\frac{3}{4}+\frac{3}{4}}{2}=\frac{3}{4}.$$

XI. General Case

General idea. The above computations can be generalized as follows.

In general, for a linearly ordered case when

$$1 \prec 2 \prec \ldots \prec k,$$

we get $s_i=\frac{i}{k+1}$; see, e.g., [1], [2], [3].

So, in general, for a partial order $\prec$, the value s_i is equal to

$$s_i=\frac{r_i}{k+1},$$

where r_i is the average value of the rank of the element i in all the linear orders which are consistent with the given partial order.

Examples. The above computations for the two options are two examples of using this general idea.

Another example is when we have $1 \prec 2$, $1 \prec 3$, and $1 \prec 4$. In this case, we have 6 possible orders:

- an order in which $1 \prec 2 \prec 3 \prec 4$;
- an order in which $1 \prec 3 \prec 4 \prec 2$;
- an order in which $1 \prec 4 \prec 2 \prec 3$;
- an order in which $1 \prec 4 \prec 3 \prec 2$;
- an order in which $1 \prec 3 \prec 2 \prec 4$;
- an order in which $1 \prec 2 \prec 4 \prec 3$.

Here, the element 1 always has rank 1, so $r_1 = 1$. The average rank of each of the elements 2, 3, and 4 is

$$\frac{2+3+4+2+3+4}{6}=3,$$

thus

$$s_1=\frac{1}{5}\text{ and } s_2=s_3=s_4=\frac{3}{5}.$$

In general, when we have $1 \prec 2$, ..., $1 \prec k$, then $r_1 = 1$ and

$$r_i=\frac{2+3+\ldots+k}{k-1}=\frac{\frac{k\cdot(k+1)}{2}-1}{k-1}=\frac{k^2+k-2}{2\cdot(k-1)}=$$

$$\frac{(k+2)\cdot(k-1)}{2\cdot(k-1)}=\frac{k+2}{2}.$$

Thus, here:

$$s_1=\frac{1}{k+1}$$

and

$$s_1=\ldots=s_k=\frac{k+2}{2\cdot(k+1)}=\frac{1}{2}\cdot\left(1+\frac{1}{k+1}\right).$$

XII. From Number-Valued to Interval-Valued Degrees

Need for interval-valued degrees. Even those experts who can describe their degrees in numerical terms usually have trouble providing an exact numerical degree: indeed, we do not have a feeling of difference between, say, degree 0.5 and degree 0.501. From this viewpoint, it is more adequate to describe degrees not by numbers but by intervals – i.e., subintervals of the interval $[0, 1]$; see, e.g., [10], [11], [12].

It is therefore desirable to transform partial orders not into numbers, but into such intervals.

Same idea works for interval-valued degrees: example. For example, suppose that we have two degrees 1 and 2 for which $1 \prec 2$. We want to assign to each of them an interval $[\underline{s}_1, \overline{s}_1]$ and $[\underline{s}_2, \overline{s}_2]$. A natural way to describe that $1 \prec 2$ is to require that $\underline{s}_1 < \underline{s}_2$ and $\overline{s}_1 < \overline{s}_2$.

Thus, we need to generate four numbers $\underline{s}_1$, $\overline{s}_1$, $\underline{s}_2$, and $\overline{s}_2$ for which $\underline{s}_1 < \overline{s}_1$, $\underline{s}_1 < \underline{s}_2$, $\overline{s}_1 < \overline{s}_2$, and $\underline{s}_2 < \overline{s}_2$. If we denote the corresponding bounds by 1^-, 1^+, 2^-, and 2^+, then we get the following partial order: $1^- \prec 1^+$, $1^- \prec 2^-$, $1^+ \prec 2^+$, and $2^- \prec 2^+$. The only two degrees for which we have no ordering relation are 1^+ and 2^-. Thus, here we have two possible linear orders:

- a linear order in which $1^- \prec 1^+ \prec 2^- \prec 2^+$, and
- a linear order in which $1^- \prec 2^- \prec 1^+ \prec 2^+$.

Here, for the average ranks, we have $r_{1^-} = 1$,

$$r_{1^+} = r_{2^-} = \frac{2+3}{2} = 2.5,$$

and $r_{2^+} = 4$, thus

$$\underline{s}_1 = \frac{1}{5}, \quad \overline{s}_1 = \underline{s}_2 = \frac{1}{2}, \quad \overline{s}_2 = \frac{4}{5}.$$

General case. We can perform similar computations for any other partially ordered set.

XIII. Remaining Open Problem

How to speed up computations? The above algorithm works OK, but for a large number of degrees k, we may have exponentially many possible linear orders, which makes the computation of the average ranks r_i taking too much time.

To effectively deal with such situations, it is desirable to come up with a more efficient algorithm for computing the average ranks r_i.

Maybe we can compute r_i by using an appropriate Monte-Carlo method – or an appropriate metaheuristic method?

What if we have the opinions of several experts? what is some experts are inconsistent? It is desirable to extend the above procedure to realistic situations when several experts describe different partial orders – and/or situations when one or more experts are inconsistent, i.e., provide "ordering" which is, in general, not transitive.

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