Which Point From an Interval Should We Choose?

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***Abstract*—In many practical situations, we know the exact form of the objective function, and we know the optimal decision corresponding to each value of the corresponding parameter x. What should we do if we do not know the exact value of x, and instead, we only know x with uncertainty – e.g., with interval uncertainty? In this case, a reasonable idea is to select one value from the given interval, and to use the optimal decision corresponding to the selected value. But which value should we choose? In this paper, we provide a solution to this problem for the situation in the simplest 1-D case. Somewhat surprisingly, it turns out the usual practice of selecting the midpoint is rarely optimal, a better selection is possible.**

I. Formulation of the Practical Problem

Often, we know the ideal-case solution. One of the main objectives of science and engineering is to provide an optimal decision in different situations. In many practical situations, we have an algorithm that provides an optimal decision based under the condition that we know the exact values of the corresponding parameters x.

In practice, we need to take uncertainty into account. In practice, we usually know x with some uncertainty. For example, often, we only know an interval $[\underline{x}, \overline{x}]$ that contains the actual (unknown) value x; see, e.g., [8].

A problem. In the case of interval uncertainty, we can implement decisions corresponding to different values $x \in [\underline{x}, \overline{x}]$. Which value should we choose?

Often, practitioners select the midpoint, but is this selection the best choice?

These are the questions that we answer in this paper.

II. Formulation of the Problem in Precise Terms

Decision making: a general description. In general, we need to make a decision u based on the state x of the system. According to decision theory, a rational person selects a decision that maximizes the value of an appropriate function known as utility; see, e.g., [1], [5], [6], [9].

We will consider situations when or each state x and for each decision u, we know the value of the utility $f(x, u)$ corresponding to us choosing u. Then, an optimal decision $u_{\rm opt}(x)$ corresponding to the state x is the decision for which this utility is the largest possible:

$$f(x, u_{\rm opt}(x)) = \max_u f(x, u). \qquad (1)$$

Decision making under interval uncertainty: 1-D case. In practice, we rarely know the exact state of the system, we usually know this state with some uncertainty. Often, we do not know the probabilities of different possible states x, we only know the bounds on different parameters describing the state.

In this paper, we will consider the simplest case:

- when the state is characterized by a single parameter, i.e., when x is a real number, and
- when a decision is also described by a single number u.

In this case, the bounds means that instead of knowing the exact state x, we only know the bounds $\underline{x}$ and $\overline{x}$ on the state, i.e., we only know that the actual (unknown) state belongs to the interval $[\underline{x}, \overline{x}]$. The question is: what decision u should we make in this case?

We also assume that the uncertainty with which we know x is relatively small, so in the corresponding Taylor series, we can only the first few terms in terms of this uncertainty.

Since we already know ho to compute the optimal value $u_{\rm opt}(x)$ corresponding to a given state x, it may be easier, instead of coming up with a new algorithm that describes u as a function of the bounds $\underline{x}$ and $\overline{x}$, to come up with a value s for which $u = u_{\rm opt}(s)$.

Decision making under interval uncertainty: towards a precise formulation of the problem. Because of the uncertainty with which we know x, for each possible decision u, we do not know the exact value of the utility, we only know that this utility is equal to $f(x, u)$ for some $x \in [\underline{x}, \overline{x}]$. Thus, all we know is that this utility value belongs to the interval

$$\left[\min_{x\in[\underline{x},\overline{x}]} f(x, u), \max_{x\in[\underline{x},\overline{x}]} f(x, u)\right]. \qquad (2)$$

According to decision theory (see, e.g., [2], [4], [5]), if for every action a, we only know the interval $[f^-(a), f^+(a)]$ of possible values of utility, then we should select the action for which the following combination takes the largest possible value:

$$\alpha \cdot f^+(a) + (1 - \alpha) \cdot f^-(a), \qquad (3)$$

where the parameter $\alpha \in [0, 1]$ describes the decision maker's degree of optimism-pessimism:

- the value $\alpha = 1$ means that the decision maker is a complete optimist, only taking into account the best-case situations,

- the value $\alpha = 0$ means that the decision maker is a complete pessimist, only taking into account the worst-case situations, and
- intermediate value $\alpha \in (0,1)$ means that the decision maker takes into account both worst-case and best-case scenarios.

Resulting formulation of the problem. In these terms our goal is:

- given the function $f(x,u)$ and the bounds $\underline{x}$ and $\overline{x}$,
- to find the value u for which the following objective function takes the largest possible value:

$$\alpha \cdot \max_{x\in[\underline{x},\overline{x}]} f(x,u) + (1-\alpha)\cdot \min_{x\in[\underline{x},\overline{x}]} f(x,u) \to \max_u . \quad (4)$$

Comment. Alternatively, we need to find s for which $u = u_{\rm opt}(s)$ maximizes the objective function (4).

III. Analysis of the Problem

We assumed that the uncertainty is small, and that in the corresponding Taylor expansions, we can keep only a few first terms corresponding to this uncertainty. Therefore, it is convenient to describe this uncertainty explicitly.

Let us denote the midpoint $\dfrac{\underline{x}+\overline{x}}{2}$ of the interval $[\underline{x},\overline{x}]$ by x_0. Then, each point x from this interval can be represented as $x = x_0 + \Delta x$, where we denoted $\Delta x \stackrel{\rm def}{=} x - x_0$. The range of possible values of Δx is $[\underline{x}-x_0, \overline{x}-x_0] = [-\Delta, \Delta]$, where we denoted $\Delta \stackrel{\rm def}{=} \dfrac{\overline{x}-\underline{x}}{2}$.

The difference Δx is small, so we should be able to keep only the few first terms in Δx.

When x is known exactly, the optimal decision is $u_{\rm opt}(x)$. Since uncertainty is assumed to be small, the optimal decision u under interval uncertainty should be close to the optimal decision $u_0 \stackrel{\rm def}{=} u_{\rm opt}(x_0)$ corresponding to the midpoint. So, the difference $\Delta u \stackrel{\rm def}{=} u - u_0$ should also be small. In terms of Δu, the original value u has the form $u = u_0 + \Delta u$. Substituting $x = x_0 + \Delta x$ and $u = u_0 + \Delta u$ into the expression $f(x,u)$ for the utility, and keeping only linear and quadratic terms in this expansion, we conclude that

$$f(x,u) = f(x_0+\Delta x, u_0+\Delta u) =$$
$$f(x_0,u_0) + f_x\cdot\Delta x + f_u \cdot \Delta u +$$
$$\frac{1}{2}\cdot f_{xx}\cdot(\Delta x)^2 + f_{xu}\cdot \Delta x\cdot\Delta u + \frac{1}{2}\cdot f_{uu}\cdot(\Delta u)^2, \quad (5)$$

where we denoted

$$f_x \stackrel{\rm def}{=} \frac{\partial f}{\partial x}(x_0,u_0), \quad f_u \stackrel{\rm def}{=} \frac{\partial f}{\partial u}(x_0,u_0),$$
$$f_{xx} \stackrel{\rm def}{=} \frac{\partial^2 f}{\partial x^2}(x_0,u_0), \quad f_{xu} \stackrel{\rm def}{=} \frac{\partial^2 f}{\partial x \partial u}(x_0,u_0),$$
$$f_{uu} \stackrel{\rm def}{=} \frac{\partial^2 f}{\partial u^2}(x_0,u_0).$$

To find an explicit expression for the objective function (4), we need to find the maximum and the minimum of this objective function when u is fixed and $x \in [\underline{x},\overline{x}]$, i.e., when $\Delta x \in [-\Delta, \Delta]$. To find the maximum and the minimum of a function of an interval, it is useful to compute its derivative. For the objective function (5), we have

$$\frac{\partial f}{\partial x} = f_x + f_{xx}\cdot\Delta x + f_{xu}\cdot\Delta u. \quad (6)$$

In general, the value f_x is different from 0; we will ignore a possible degenerate case when $f_x = 0$. Since we assumed that the differences Δx and Δu are both small, a linear combination of these two differences is smaller than $|f_x|$. Thus, on the whole interval $\Delta x \in [-\Delta,\Delta]$, the sign of the derivative $\dfrac{\partial f}{\partial x}$ is the same as the sign $s_x \stackrel{\rm def}{=} {\rm sign}(f_x)$ of the value f_x.

Hence:

- when $f_x > 0$ and $s_x = +1$, then the function $f(x,u)$ is an increasing function of x; its maximum is attained when x is attained its largest possible values $\overline{x}$, i.e., when $\Delta x = \Delta$, and its minimum is attained when $\Delta x = -\Delta$;
- when $f_x < 0$ and $s_x = -1$, then the function $f(x,u)$ is an decreasing function of x; its maximum is attained when x is attained its smallest possible values $\underline{x}$, i.e., when $\Delta x = -\Delta$, and its minimum is attained when $\Delta x = \Delta$.

In both cases, the maximum of the utility function $f(x,u)$ is attained when $\Delta x = s_x\cdot\Delta$ and its minimum is attained when $\Delta x = -s_x\cdot\Delta$. Thus,

$$\max_{x\in[\underline{x},\overline{x}]} f(x,u) = f(x_0 + s_x\cdot\Delta, u_0+\Delta u) =$$
$$f(x_0,u_0) + f_x\cdot s_x\cdot\Delta + f_u\cdot\Delta u +$$
$$\frac{1}{2}\cdot f_{xx}\cdot(\Delta)^2 + f_{xu}\cdot s_x\cdot\Delta\cdot\Delta u + \frac{1}{2}\cdot f_{uu}\cdot(\Delta u)^2, \quad (7)$$

and

$$\min_{x\in[\underline{x},\overline{x}]} f(x,u) = f(x_0 - s_x\cdot\Delta, u_0+\Delta u) =$$
$$f(x_0,u_0) - f_x\cdot s_x\cdot\Delta + f_u\cdot\Delta u +$$
$$\frac{1}{2}\cdot f_{xx}\cdot(\Delta)^2 - f_{xu}\cdot s_x\cdot\Delta\cdot\Delta u + \frac{1}{2}\cdot f_{uu}\cdot(\Delta u)^2. \quad (8)$$

Therefore, our objective function (4) takes the form

$$\alpha\cdot\max_{x\in[\underline{x},\overline{x}]} f(x,u) + (1-\alpha)\cdot\min_{x\in[\underline{x},\overline{x}]} f(x,u) =$$
$$f(x_0,u_0) + (2\alpha-1)\cdot f_x\cdot s_x\cdot\Delta + f_u\cdot\Delta u +$$
$$\frac{1}{2}\cdot f_{xx}\cdot(\Delta)^2 + (2\alpha-1)\cdot f_{xu}\cdot s_x\cdot\Delta\cdot\Delta u + \frac{1}{2}\cdot f_{uu}\cdot(\Delta u)^2. \quad (9)$$

To find the optimal value $\Delta u_{\max} = u - u_0$ for which the objective function (4) attains its largest possible value, we differentiate the expression (9) for the objective function (4) with respect to u and equate the derivative to 0. As a result, we get:

$$f_u + (2\alpha-1)\cdot f_{xu}\cdot s_x\cdot\Delta + f_{uu}\cdot\Delta u_{\max} = 0, \quad (10)$$

i.e.,

$$\Delta u_{\max} = -\frac{f_u + (2\alpha - 1)\cdot f_{xu}\cdot s_x\cdot \Delta}{f_{uu}}. \quad (11)$$

To simplify this expression, let us now take into account that for each x, the function $f(x,u)$ attains its maximum at the known value $u_{\rm opt}(x)$. Differentiating expression (5) with respect to u and equating the derivative to 0, we get:

$$f_u + f_{xu}\cdot \Delta x + f_{uu}\cdot \Delta u = 0. \quad (12)$$

For $x = x_0$, i.e., when $\Delta x = 0$, this maximum is attained when $u = u_0$, i.e., when $\Delta u = 0$. Substituting $\Delta x = 0$ and $\Delta u = 0$ into the formula (12), we conclude that $f_u = 0$, and thus, the formula (11) takes a simplified form

$$\Delta u_{\max} = -\frac{(2\alpha - 1)\cdot f_{xu}\cdot s_x\cdot \Delta}{f_{uu}}. \quad (13)$$

In general, we can similarly expand $u_{\rm opt}(x)$ in Taylor series and keep only a few first terms in this expansion:

$$u_{\rm opt}(x) = u_{\rm opt}(x_0 + \Delta x) = u_0 + u_x\cdot \Delta x, \quad (14)$$

where we denoted $u_x \stackrel{\rm def}{=} \dfrac{\partial u_{\rm opt}}{\partial x}$. Thus, for the optimal decision, $\Delta u = u_{\rm opt}(x) - u_0 = u_x\cdot \Delta x$. Substituting this expression and $f_u = 0$ into the formula (12), we conclude that

$$f_{xu}\cdot \Delta x + f_{uu}\cdot u_x\cdot \Delta x = 0$$

for all Δx. Thus, $f_{xu} + f_{uu}\cdot u_x = 0$, and

$$\frac{f_{xu}}{f_{uu}} = -u_x. \quad (15)$$

Substituting the expression (15) into the formula (13), we conclude that

$$\Delta u_{\max} = (2\alpha - 1)\cdot u_x\cdot s_x\cdot \Delta. \quad (16)$$

Let us describe this solution in terms of the value $s \in [\underline{x}, \overline{x}]$ for which $u(s)$ is equal to the optimal value $u_0 + \Delta u_{\max}$. Since $s \in [\underline{x}, \overline{x}]$, we can represent s as $s = x_0 + \Delta s$, where $\Delta s \stackrel{\rm def}{=} s - x_0$. Thus,

$$u(s) = u(x_0 + \Delta s) = u_0 + u_x\cdot \Delta s. \quad (17)$$

Equating this expression and the desired value $u_0 + \Delta u_{\max}$ and using the expression (16) for $\Delta u_{\max}$, we conclude that

$$u_x\cdot \Delta s = (2\alpha - 1)\cdot u_x\cdot s_x\cdot \Delta, \quad (18)$$

and thus,

$$\Delta s = (2\alpha - 1)\cdot s_x\cdot \Delta. \quad (19)$$

Here, s_x is the sign of the derivative f_x. We have two options:

- If $f_x > 0$, i.e., if the objective function increases with x, then $s_x = 1$, and the formula (19) takes the form $\Delta s = (2\alpha - 1)\cdot \Delta$. In this case, by using the expressions for x_0 and Δ in terms of $\overline{x}$ and $\underline{x}$, we get:

$$s = x_0 + \Delta s = \frac{\underline{x} + \overline{x}}{2} + (2\alpha - 1)\cdot \frac{\overline{x} - \underline{x}}{2} =$$

$$\alpha\cdot \overline{x} + (1 - \alpha)\cdot \underline{x}. \quad (20)$$

- If $f_x < 0$, i.e., if the objective function decreases with x, then $s_x = -1$, and the formula (19) takes the form $\Delta s = -(2\alpha - 1)\cdot \Delta$. In this case,

$$s = x_0 + \Delta s = \frac{\underline{x} + \overline{x}}{2} - (2\alpha - 1)\cdot \frac{\overline{x} - \underline{x}}{2} =$$

$$\alpha\cdot \underline{x} + (1 - \alpha)\cdot \overline{x}. \quad (21)$$

So, we arrive at the following recommendation.

IV. Solution to the Problem

Formulation of the problem: reminder. We assume that we know the objective function $f(x,u)$ that characterizes our gain in a situation when the actual value of the parameter is x and we select an alternative u.

We also assume that for every value x, we know the optimal value $u_{\rm opt}(x)$ for which the objective function attains its largest possible value.

In a practical situation in which we only know that the value x is contained in an interval $[\underline{x}, \overline{x}]$, we need to select some value $s \in [\underline{x}, \overline{x}]$, and then select the alternative $u_{\rm opt}(s)$ corresponding to the selected value s.

Description of the solution. The solution to our problem depends on whether the objective function $f(x,u)$ is an increasing or decreasing function of the parameter x.

If the objective function is an increasing function of x, then we should select a solution corresponding to

$$x = \alpha\cdot \overline{x} + (1 - \alpha)\cdot \underline{x}, \quad (22)$$

where α is the optimism-pessimism parameter that characterizes the decision maker.

If the objective function is a decreasing function of x, then we should select a solution corresponding to

$$x = \alpha\cdot \underline{x} + (1 - \alpha)\cdot \overline{x}. \quad (23)$$

Comment. Thus, the usual selection of the midpoint s is only optimal for decision makers for which $\alpha = 0.5$; in all other cases, this selection is *not* optimal.

Discussion. Intuitively, the above solution is in good accordance with the Hurwicz criterion:

- when the objective function increases with x, the best possible situation corresponds to $\overline{x}$, and the worst possible situation corresponds to $\underline{x}$; thus, the Hurwicz combination corresponds to the formula (22);
- when the objective function increases with x, the best possible situation corresponds to $\underline{x}$, and the worst possible situation corresponds to $\overline{x}$; thus, the Hurwicz combination corresponds to the formula (23).

This intuitive understanding is, however, not a proof – Hurwicz formula combines utilities, not parameter values.

V. Future Work

We have provided a solution for the simplest case, when we have only one parameter x describing the system and only one parameter u describing possible alternatives. It is desirable to extend our solution to the case when we have several parameters x and several parameters u.

What if, in addition to the interval, we also have partial information about the probabilities of different values x from this interval?

It is also desirable to extend this solution to situations when instead of the exact interval, we only have fuzzy information about x [3], [7], [10] – i.e., in terms of α-cuts, we have different intervals $[\underline{x}, \overline{x}]$ for different levels of certainty α.

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