The Range of a Continuous Functional Under Set-Valued Uncertainty Is Always an Interval*

Vladik Kreinovich and Olga Kosheleva
University of Texas at El Paso, 500 W. University,
El Paso, TX 79968, USA
vladik@utep.edu,olgak@utep.edu

Abstract

One of the main problems of interval computations is computing the range of a given function on a given multi-D interval (box). It is known that the range of a continuous function on a box is always an interval. However, if, instead of a box, we consider the range over a subset of this box, the range is, in general, no longer an interval. In some practical situations, we are interested in computing the range of a functional over a function defined with interval (or, more general, set-valued) uncertainty. At first glance, it may seem that under a non-interval set-valued uncertainty, the range of the functional may be different from an interval. However, somewhat surprisingly, we show that for continuous functionals, this range is always an interval.

Keywords: interval computations, interval mathematics, continuous functional, set-valued uncertainty, range of a functional
AMS subject classifications: 65G20, 65G40, 46E99

1 Formulation of the Problem

Computing the range over a multi-D interval (box): reminder. In many practical situations, we know the dependence $y = f(x_1, \ldots, x_n)$ between the desired quantity y and the easy-to-measure quantities $x_1, \ldots, x_n$. In the ideal case, when we know the exact values $x_1, \ldots, x_n$ of the corresponding quantities, we can use this dependence to compute the value of the desired quantity y.

In practice, measurements are never absolutely exact; see, e.g., [3]: the measurement result $\widetilde{x}_i$ is, in general, somewhat different from the actual (unknown) value x_i of the corresponding quantity. In many cases, the only information that we have about the corresponding measurement error $\Delta x_i \stackrel{\text{def}}{=} \widetilde{x}_i - x_i$ is the upper bound Δ_i on its absolute value: $|\Delta x_i| \le \Delta_i$. In such cases, once we get the measurement result $\widetilde{x}_i$, the only information that we have about the actual value x_i is that this value is somewhere on the interval $[\underline{x}_i, \overline{x}_i]$, where $\underline{x}_i = \widetilde{x}_i - \Delta_i$ and $\overline{x}_i = \widetilde{x}_i + \Delta_i$.

*Submitted: August 7, 2016; Revised: April 23, 2017; Accepted:

Different combinations of possible values $(x_1, \ldots, x_n)$ lead, in general, to different values of $y = f(x_1, \ldots, x_n)$. It is therefore desirable to find the range of such values of y, i.e., the set

$$\{f(x_1, \ldots, x_n) : x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, x_n \in [\underline{x}_n, \overline{x}_n]\}.$$

Computing this range is one of the main problem of *interval computations*; see, e.g., [1, 2].

It is well known that for a continuous function $f(x_1, \ldots, x_n)$, the resulting range is always an interval.

From intervals to more general sets. Sometimes, in addition to knowing the bounds $\underline{x}_i$ and $\overline{x}_i$, we also know that some values from the corresponding interval $[\underline{x}_i, \overline{x}_i]$ are not possible. In such cases, the set X_i of all possible values of each quantity x_i is a proper subset of an interval – and often, a subset which is not connected.

For example, if we measure the kinetic energy of a particle moving in the x-direction, we then know the absolute value of its velocity, but not its direction. In this example, the set of all possible values of the velocity consists of two value $X = \{-v, v\}$. If we take into account that the energy – and thus, the absolute value of the velocity – can only be measured with some accuracy, then we get a more realistic set $X = [-\overline{v}, -\underline{v}] \cup [\underline{v}, \overline{v}]$.

In such cases, the range $\{f(x_1, \ldots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n\}$ is not necessarily an interval. For example, if each set X_i consists of finitely many points, then we have finitely many possible tuples $(x_1, \ldots, x_n)$ and thus, only finitely many values $f(x_1, \ldots, x_n)$ corresponding to these tuples – so the range is a finite set and hence not an interval.

Continuous case. In some practical situations, the desired quantity y depends not on finitely many quantities $x_1, \ldots, x_n$, but on the whole signal $x(t)$, i.e., in effect, on infinitely many values $x(t)$ corresponding to all possible moments of time t from some interval $\left[\underline{T}, \overline{T}\right]$. In other words, we have a *functional* $y = f(x)$ that describes how the value y depends on the signal $x(t)$.

For example, one way to find the location on a submerged submarine is to measure its acceleration $x(t)$. If we know the initial velocity $v(\underline{T})$, then the velocity at each moment t can be found by integrating the acceleration, as $v(t) = v_0 + \int_{\underline{T}}^{t} x(s)\,ds$. Thus, once we know the initial coordinate y_0, we can find the coordinate y at the current moment $\overline{T}$ as an integral

$$y = y_0 + \int_{\underline{T}}^{\overline{T}} v(t)\,dt = y_0 + \int_{\underline{T}}^{\overline{T}} \left(v_0 + \int_{\underline{T}}^{t} x(s)\,ds\right)\,dt.$$

The values $x(t)$ can only be measured with some uncertainty. Thus, for each t, instead of the exact value $x(t)$, we only know the interval $[\underline{x}(t), \overline{x}(t)]$ that contains the actual (unknown) value $x(t)$. For different functions $x(t)$ from this interval, in general, we have different values of the function $f(x)$. It is therefore desirable to find the range of the functional $f(x)$ under this interval uncertainty, i.e., the range

$$\{f(x) : \underline{x}(t) \le x(t) \le \overline{x}(t) \text{ for all } t\}.$$

What if we have set uncertainty in the continuous case: formulation of the problem. What if for each t, in addition to the interval $[\underline{x}(t), \overline{x}(t)]$, we also know that the actual values $x(t)$ can only belong to an appropriate subset $X(t)$ of this interval? What can we then say about the range

$$\{f(x) : x(t) \in X(t) \text{ for all } t\}?$$

At first glance, it may seem that, similarly to the usual set-valued case, we can have a non-interval range. However, as we show in the paper, in the continuous case, the range is *always* an interval – even when we have set uncertainty with non-connected sets $X(t)$ instead of interval uncertainty.

2 Main Result

Proposition.

- *Let* $\left[\underline{T}, \overline{T}\,\right]$ *be an interval.*
- *Let* $\Delta > 0$ *be a real number.*
- *Let* X *be a mapping that maps each moment* $t \in \left[\underline{T}, \overline{T}\,\right]$ *into a subset*

$$X(t) \subseteq [-\Delta, \Delta].$$

- *Let* f *be a functional that maps every measurable function* $x(t)$ *– for which* $|x(t)| \le \Delta$ *for all* t *– into a real number.*
- *We also assume that for some* $p > 0$*,* f *is continuous in terms of the* L^p*-distance*

$$d(x_1, x_2) = \left(\int |x_1(t) - x_2(t)|^p \, dt\right)^{1/p}.$$

Under these assumptions, the range $\{f(x) : x(t) \in X(t) \text{ for all } t\}$ *is a connected set.*

Comment. On the real line, the only connected sets are intervals – finite or infinite, open or closed or semi-open, degenerate or non-degenerate. Thus, the above results says that the range of a continuous functional under set-valued uncertainty is always an interval.

Proof. To prove connectedness, we must prove that for every two measurable functions $x_1(t)$ and $x_2(t)$, each real values y between $f(x_1)$ and $f(x_2)$ can also be represented as $f(x)$ for some measurable function $x(t)$ for which $x(t) \in X(t)$ for all t.

Indeed, let us consider, for each value $s \in \left[\underline{T}, \overline{T}\,\right]$, an auxiliary function $x_{(s)}(t)$ which is defined as follows:

- for $t \le s$, we have $x_{(s)}(t) = x_1(t)$; and
- for $t > s$, we have $x_{(s)}(t) = x_2(t)$.

It is easy to see that each of these auxiliary functions is also measurable.

For each t, the value $x_{(s)}(t)$ is equal to either $x_1(t)$ or $x_2(t)$. Both values are contained in the set $X(t)$, so we can conclude that $x_{(s)}(t) \in X(t)$ for all moments t.

From the above definition of the function $x_{(s)}$, it follows that:

- for $s = \overline{T}$, we have $x_{(s)} = x_1$, and
- for $s = \underline{T}$, we have $x_{(s)} = x_2$.

For every two numbers $s < s'$, the values of the functions $x_{(s)}(t)$ and $x_{(s')}(t)$ differ only for $t \in [s, s']$, where one of them is equal to $x_1(t)$ and another one to $x_2(t)$. Since the values of both functions $x_1(t)$ and $x_2(t)$ are located on the interval $[-\Delta, \Delta]$, the difference $|x_1(t) - x_2(t)|$ cannot exceed 2Δ. Thus, we have:

$$d^p\left(x_{(s)}, x_{(s')}\right) = \int_{\underline{T}}^{\overline{T}} \left|x_{(s)}(t) - x_{(s')}(t)\right|^p \, dt = \int_s^{s'} |x_1(t) - x_2(t)|^p \, dt \le$$

$$(s' - s) \cdot (2\Delta)^p.$$

As the difference $|s - s'|$ decreases, the distance $d\left(x_{(s)}, x_{(s')}\right)$ tends to 0. Thus, the mapping $s \to x_{(s)}$ is continuous in the L^p-metric.

Since the functional $f(x)$ is continuous in the sense of this metric, we can therefore conclude that the mapping $s \to f\left(x_{(s)}\right)$ is also continuous. A continuous functions from real numbers to real numbers attains, with every two values, all intermediate values as well. Thus, for every real number y between the values $f(x_1) = f\left(x_{(\overline{T})}\right)$ and $f(x_2) = f\left(x_{(\underline{T})}\right)$, there exists a value s for which $f\left(x_{(s)}\right) = y$. Since we have shown that $x_{(s)}(t) \in X(t)$ for each t, this means that y indeed belongs to the desired range, thus the range is indeed connected.

The proposition is proven.

Acknowledgments

This work was supported in part by the National Science Foundation grants CAREER 0953339, HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and DUE-0926721, and by an award “UTEP and Prudential Actuarial Science Academy and Pipeline Initiative” from Prudential Foundation.

The authors are thankful to the anonymous referees for valuable suggestions.

References

[1] L. Jaulin, M. Kiefer, O. Dicrit, and E. Walter, *Applied Interval Analysis*, Springer, London, 2001.

[2] R. E. Moore, R. B. Kearfott, and M. J. Cloud, *Introduction to Interval Analysis*, SIAM, Philadelphia, 2009.

[3] S. G. Rabinovich, *Measurement Errors and Uncertainty: Theory and Practice*, Springer Verlag, Berlin, 2005.