Concepts of solutions of uncertain equations with intervals, probabilities and fuzzy sets for applied tasks

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***Abstract*—The focus of this paper is to clarify the concepts of solutions of linear equations in interval, probabilistic, and fuzzy sets setting for real world tasks. There is a fundamental difference between formal definitions of the solutions and physically meaningful concepts of solution in applied tasks, when equations have uncertain components. For instance, a formal definition of the solution in terms of Moore interval analysis can be completely irrelevant for solving a real world task. We show that formal definitions must follow a meaningful concept of the solution in the real world. The contribution of this paper is the seven formalized definitions of the concept of solution for the linear equations with uncertain components in the interval settings that are interpretable in the real world tasks. It is shown that these definitions have analogies in probability and fuzzy set terms too. These new formalized concepts of solutions are generalized for difference and differential equations under uncertainty.**

Keywords—interval equations, fuzzy equations, stochastic equations, quantifiers, difference equation, differential equations.

1. Introduction

Granular Computing (Pedrycz, Chen, 2011, 2015) approach focuses on analyzing data at multiple levels of details or scales in a variety of granulation settings (Dubois, Prade, 2016; Xu, Wang, 2016). Often each level has its own uncertainties and needs specific methods for modeling such uncertainties (Kovalerchuk, Perlovsky, 2012). Granular computing is an integral part of Computational Intelligence (Pedrycz, Chen, 2011, 2015), Big Data (Pedrycz, Chen, 2015) and Decision Making (Pedrycz, Chen, 2015). In all these domains solving equations under uncertainty is common part of many tasks. This includes equations with parameters represented by probabilities, intervals or fuzzy sets.

At first glance, it seems reasonable to translate the concept of solution for equations without uncertainty to the corresponding type of uncertainty. This is how many researchers and practitioners solve the corresponding uncertain problems. This paper stresses that the resulting solutions are sometimes inadequate, because to come up with a correct solution, we need to analyze the original problem under uncertainty, not only its exact prototype. Often, for exact data, different practical problems lead to the same solution, while *in the presence of uncertainty* these problems often lead to *completely different solutions* as we show. This ambiguity cannot be resolved by simply modifying the usual formal approach: e.g., even for the simple case, when the exact-case solution is just a subtraction, in the uncertainty case, there can be multiple different definitions of a solution.

Most of the work in fuzzy equations has been concerned with algorithms and theorems for solving fuzzy linear equations, e.g., (Yager, 1979; Sanchez, 1984; Di Nola, Pedrycz, Sessa, 1985, R. Zhao and R. Govind, 1991; Peeva, 1991, Buckley, Feuring, 2000; Horcık, 2007; Skalna et al, 2008, Allahviranloo et al, 2011, Ghomashi et al, 2014). Zadeh's **Extension Principle (EP)** (Zadeh, 1975; Yager, 1986) in exact or approximate settings was a common assumption in many of these studies. The typical method to solve a fuzzy equation is converting to a set of interval tasks with alpha-cuts (e.g., Skalna et al, 2008). The paper by Buckley and Feuring (2000) that is representative for this type of studies is analyzed in section 5.

One work is standing out (Dubois, Prade, 1984) where the authors provided the arguments and an example that show the need to go beyond the extension principle.

The concepts of *optimistic* and *pessimistic operations* on fuzzy numbers have been proposed in this context. We use these productive concepts with some modifications to define the respective concepts of the optimistic and pessimistic solutions for uncertain equations.

The focus of this paper – which largely expands on (Kreinovich, 2016) -- is to clarify the concepts of solutions of linear equations in interval, probabilistic, and fuzzy-set setting for real-world tasks. When equations have uncertain components in applied tasks, there is a fundamental difference between formal definitions of the solutions and a physically meaningful concept of a solution. For instance, a formal definition of the solution in terms of Moore interval analysis can be completely irrelevant for solving a real-world task. We show that formal definitions must follow a meaningful concept of the solution in the real world, not another way around.

The solution that is claimed to be a solution of the uncertain equation should be a *solution of the real-world problem* not a just a result of a formal mathematical definition. In line with this, the Pythagorean Theorem survived for 2000 years because it solved the real-world problems, not just because it discovered an elegant mathematical property of some mathematical definitions. Hisdal (1988) worded this *challenge* in the following way: somebody proposed a solution, now we need to find a problem for it.

The example of equation with uncertainty is

$$A+X=B \quad (1)$$

where A, X and B are intervals, pdfs, or fuzzy sets. The task is to find X. The sum of multiple dependent intervals, pdfs and fuzzy sets is not obvious. The same is true for the pdf and fuzzy membership function for the sum. For instance, a sum of $x+y$ with respective pdfs *ill-posed* f(x) and g(y) has a joint probability distribution $q(x,y)=f(x)\cdot g(y|x)$. If f and g are independent then $q(x,y)=f(x)\cdot g(y)$. However in general we cannot make this independence assumption, and the conditional probability $g(y|x)$ often is unknown, As a result equation (1) is not fully defined to be solved. In the fuzzy set setting it is related to the selection of the aggregation operator.

The contribution of this paper is the seven new formalized definitions of the concepts of solutions for the linear equations with uncertain components in the interval settings that are interpretable in the real world tasks. These definitions convert ill-posed problems into complete mathematical tasks using the clearly interpretable concept of optimistic and pessimistic solutions. It is shown that these definitions have analogies in probability and fuzzy set terms too, and have generalizations for difference and differential equations under uncertainty.

2. TASKS

2.1 Airport Task

Dubois and Prade (1984) formulated an applied example (in fuzzy set setting) where they used concepts of optimistic and pessimistic operations. Below we consider their original task, as well as our generalization to formulate and compare precise point-based, interval-based, probability-based, and fuzzy set-based formulations of what is a solution.

Airport Task. Person P wants to ensure that he will not miss a plane. His goal is to make sure he *arrives at the airport by desired time* T_A (expressed precisely or with some uncertainty as an *interval, as a probability, or a fuzzy set/number*) in spite of imprecise duration D of his preceding activities such as wake up time, washing, eating breakfast, driving to the airport, etc.). Dubois and Prade formulated the goal as finding required wakeup time.

Later we will consider a more general goal. To clarify that we need to understand the problem when formalizing what is a solution, we will also consider a modification of the above task, in which we know that the person P arrived at the airport on time, and we want to find out when he/she woke up.

2.2. Precise and Interval settings

Consider first the **precise setting** where all data are known and precise. Let precise durations of all activities be d_1, $\underline{d}_2, \ldots,$ d_n and the desired time to arrive to the airport t_A be known. Then the required wake up time t_w is trivially computed as

$$t_w = t_A - (d_1+d_2+\ldots+d_n).$$

The exact same formula can compute the wake-up time when we know when the person P arrived at the airport, and we know the durations of all the activities. For example, if we need to be at the airport at 2 pm, and the travel time from home is 20 minutes, then we need to leave home at 1:40 pm. Similarly, if we know that the person P arrived at the airport at 3 pm, and we know that it took him/her exactly 20 minutes to get there, this means that the person P left home at 2:40 pm – the same answer as for the previous problem.

Now consider the **interval setting** where t_A and all d_i are substituted by intervals, T_A and D_i. Then the required wake up time t_w is also trivially computed as

$$t_w = t_{As} - (d_{1e}+d_{2e}+\ldots+d_{ne}),$$

where t_{As} is the start point of the interval T_A, i.e., earliest desired arrival time, and each d_{ie} is the endpoint of the respective interval D_i, i.e., longest time of each activity. This solution (wake up time t_w) can be called as the *earliest pessimistic solution*. It ensures that he will catch the plane in *most pessimistic* case when longest durations of all activities will happen. It is also the *earliest* among all *pessimistic* cases because it ensures the earliest arrival to the airport within the desired arrival interval T_A. Respectively the *latest pessimistic solution* is $t_w = t_{Ae} - (d_{1e}+d_{2e}+\ldots+d_{ne})$, where t_{Ae} is the end point of the interval T_A, i.e., latest desired arrival time. All other **pessimistic solutions** are between these two earliest and latest solutions.

Note that for the modified airport task, the solution corresponding to interval uncertainty is different. For example, if we want to arrive not earlier than 2 pm, but not later than 3 pm, and the travel time takes between 20 and 40 minutes, then, to guarantee that we arrive exactly between 2 and 3 pm, we need to leave home between 1:40 pm and 2:20 pm. If we leave home before 1:40 pm, we run a risk of arriving too early (before 2 pm), and if we leave after 2:20 pm, we run a risk of arriving too late (after 3 pm). So, in this case, the solution to the original Airport task is the time interval [1.40 pm, 2.20 pm].

Let us now consider the modified task under the same interval uncertainty. Suppose that the person P arrives at the airport between 2 pm and 3 pm, and we know that the travel time from home is between 20 and 40 minutes. We want to find out when the person P left home. Based on this information, the only thing we can conclude is that P left home between 1:20 pm (= 2 pm – 40 minutes) and 2:40 pm (= 3 pm – 20 minutes). The resulting interval [1:20 pm, 2:40 pm] is exactly what interval computations predict – and it is different from the previous interval (that can be obtained, by the way, by using modal interval arithmetic; see, e.g., (Shary 1996).

The bounds corresponding to the modified task are particular examples of what we call optimistic solutions. All **optimistic solutions** are between two earliest and latest optimistic solutions,

$t_{w1}=t_{As}$ - (d_{1s}+d_{2s}+…+d_{ns}), $t_{w2}=t_{Ae}$ - (d_{1s}+d_{2s}+…+d_{ns}).

We call these solutions optimistic because they assume the shortest durations of all activities. The first one, t_{w1}, ensures the earliest optimistic wake up time t_w because it uses the start point of the interval T_A. Similarly the second one, t_{w2}, ensures the latest optimistic wake up time t_w because it uses the end point of the interval T_A.

All *other solutions* are between latest optimistic wake up time and earliest pessimistic wake up time. Note that optimistic solutions *may not be appropriate solutions* for the original task, because the person P may not be able to conduct activities with the shortest duration. For instance, traffic jam or an accident can prevent the person P from enjoying the shortest driving time.

Now we will represent concepts of interval equations and solutions in the formal terms including pessimistic and optimistic solutions using universal and existential quantifiers. Consider equation (2) where A, X and B are intervals. For the airport task X is T_w, B is T_A, and all D_i are the same as above.

$$X+D_1+D_2+\dots D_n = B \qquad (2)$$

There are four specifications of equation (2) (Sharyi, 1996, 2002; Horcık, 2008) that are different tasks:

(S1) $\forall i\ \forall d_i \in D_i\ \ \forall\ b \in B\ \exists\ x \in X\text{:}\ \ x+d_1+d_2+\dots+d_n=b$

(S2) $\forall \mathrm{i}\ \forall \mathrm{d_i} \in D_i\ \ \exists\ b \in B\ \ \exists\ x \in X\text{:}\ \ x+d_1+d_2+\dots+d_n=b$

(S3) $\forall i\ \ \exists d_i \in D_i\ \ \forall\ b \in B\ \exists\ x \in X\text{:}\ \ x+d_1+d_2+\dots+d_n=b$

(S4) $\forall i\ \ \exists d_i \in D_i\ \ \exists\ b \in \mathrm{B}\ \ \exists\ x \in X\text{:}\ \ x+d_1+d_2+\dots+d_n=b$

Pessimistic formulations

(S5) $\forall i\ d_i=d_{ie} \in D_i\ \forall b \in B\ \exists\ x \in X\text{: } x+d_1+d_2+\dots+d_n=b,$

where d_{ie} is the end point of interval $D_i=[d_{is},d_{ie}]$. All pessimistic formulations are covered by (S5).

(S6) $\forall i\ d_i=d_{ie} \in D_i\ b=b_s \in B\ \ \exists\ x \in X\text{: } x+d_1+d_2+\dots+d_n=b,,$

where $\mathrm{d_{ie}}$ is the end point of interval $\mathrm{D_i=[d_{is},d_{ie}]}$, and $\mathrm{b_s}$ is the start point of interval $\mathrm{b_s} \in \mathrm{B=[b_s,b_e]}$. The earliest pessimistic formulation is covered by (S6).

(S7) $\forall i\ d_i=d_{ie} \in D_i\ b=b_e \in B\ \ \exists\ x \in X\text{: } x+d_1+d_2+\dots+d_n=b,,$

where d_{ie} is the end point of interval $D_i=[d_{is},d_{ie}]$ and $\mathrm{b_e}$ is the start point of interval $b_s \in B=[b_s,b_e]$. The latest pessimistic formulation is covered by (S7).

Optimistic formulations

(S8) $\forall i\ d_i=d_{is} \in D_i\ \forall b \in B\ \exists\ x \in X\text{: } x+d_1+d_2+\dots+d_n=b,$

where d_{ie} is the end point of interval $D_i=[d_{is},d_{ie}]$. All optimistic formulations are covered by (S8).

(S9) $\forall i\ d_i=d_{is} \in D_i\ b=b_s \in B\ \ \exists\ x \in X\text{:}\ \ x+d_1+d_2+\dots+d_n=b,$

where d_{ie} is the end point of interval $D_i=[d_{is},d_{ie}]$ and b_s is the start point of interval $b_s \in B=[b_s,b_e]$. The earliest optimistic formulation is covered by (S9).

(S10)$\forall i\ d_i=d_{is} \in D_i\ b=b_e \in B\ \exists\ x \in X\text{:}\ \ x+d_1+d_2+\dots+d_n=b,$

where $\mathrm{d_{ie}}$ is the end point of interval $D_i=[d_{is},d_{ie}]$ and b_e is the start point of interval $b_s \in B=[b_s,b_e]$. The latest pessimistic formulation is covered by (S10).

As we can see the sum of intervals for X depends not only on X, but on B and all D_i. It can be viewed as a form of a parametric sum. For comparison see (Popova, 2013), where a parametric formulation and a solution for the interval linear equations are proposed.

Conclusion from these ten formulations for the interval equation (2):

1) In case of uncertainty, the equation (2) itself is incomplete; we need additional information to successfully solve the corresponding practical problem.
2) Each formulation S1-S10 added to Equation (2) produces a mathematically complete formulation that is sufficient for identifying whether X is a solution or not and for designing an algorithm to find the solution X.
3) Each formulation from S1-S10 of equation (2) produces its own set X of solutions.
4) There is no room for the single interval arithmetic to solve equation (2). The single interval arithmetic would produce the same X for all formulations.

As it is easy to expect the same conclusions will be true if components of equation (2) are probabilities or fuzzy sets, because intervals are the simplest forms of both of them. It was well recognized in the probability literature by noticing the need in additional information to make equation (2) mathematically complete. Examples of this information are knowledge from the *specific task*, different forms of *regularization* of the equation, and properties such as *smoothness* (see section III).

It was also recognized in the fuzzy systems literature long time ago (Dubois, Prade, 1984), but at the best of our knowledge was left mostly undeveloped. A single fuzzy arithmetic based in the Zadeh's extension principle continues to dominate in fuzzy systems literature, while new developments beyond of it also started (Piegat, Pluciński, 2015). The major message of this paper is that studies must expand beyond this narrow focus on a single type of fuzzy arithmetic, but to developing multiple fuzzy arithmetic rules derived from the real world tasks, not just postulated.

Now consider the **interval setting with vector not scalar *X***. Let, for instance each x consist of four variables that needs to be found, $x=(x_1,x_2,x_3,x_4)$. In the original airport task formulation above we assumed that the person P controls only the wake up time, x. In fact a traveler can control to some extent at least 4 variables: *wake up time*, x_1, *washing time*, x_2, *eating breakfast time*, x_3, and *departure time from home*, x_4.

Assume that the traveler can control these uncertain times within respective intervals, $X_1,...,X_4$. This means that in the equation $X+D_1+D_2+...+D_n=B$ some D_i are moving to X. As before B is the *desired arrival time at the airport* given as an interval time. This interval depends on the departure time and is rather controlled by the airport and the airlines, than by the passengers. Airlines commonly ask to be at the airport 2 hours before the departure time, t_d. Thus for this example the person P can set up a desired B as an hour length interval, B=[t_d-2.5, t_d-1.5]. In these terms in the interval formulation we need to solve the interval equation

$$X_1+X_2+X_3+X_4+D_1+D_2+...+D_k=B \quad (3)$$

Respectively task specifications S1-S10 for Eq. (3) can be rewritten. Below we show rewritten S1-S4:

(S1) $\forall i\ \forall d_i \in D_i\ \forall\ b \in B\ \forall j\ \ \exists\ x_j \in X_j$: $x_1+x_2+x_3+x_4+d_1+d_2+\ldots+d_k=b$

(S2) $\forall i\ \forall d_i \in D_i\ \ \exists\ b \in B\ \ \forall j\ \exists\ x_j \in X_j$: $x_1+x_2+x_3+x_4+d_1+d_2+\ldots+d_k=b$

(S3) $\forall i\ \ \exists d_i \in D_i\ \forall\ b \in B\ \forall j\ \ \exists\ x_j \in X_j$: $x_1+x_2+x_3+x_4+d_1+d_2+\ldots+d_k=b$

(S4) $\forall i\ \ \exists d_i \in D_i\ \ \exists\ b \in B\ \ \forall j\ \exists\ x_j \in X_j$: $x_1+x_2+x_3+x_4+d_1+d_2+\ldots+d_k=b$

The Eq. (3) presents an interesting case that links interval and probabilistic formulations. While in the interval formulation, all points in all X_i and D_j are considered as equally possible/probable their sums have different frequencies. In probabilistic formulation if all X_i and D_j represent independent uniform pdfs, then their sum will be a unimodal pdf. This means that different sums $\{b\}$ have different probabilities. Respectively some sum b_1 can have a much larger set of solution vectors $\{(x_1,x_2,x_3,x_4)\}$, than another sum b_2. Selecting b_1 with a greater probability value gives him potentially more options for wake up time, duration of breakfast and other activities that he can control. This leads to another task specification S11:

(S11) $\forall i\ \forall d_i \in D_i\ \forall\ b \in B\ \forall j\ \ \exists\ x_j \in X_j$:
$x_1+x_2+x_3+x_4+d_1+d_2+\ldots+d_k=b\ \&\ b = arg\ (max\ f(b_r))$,

where $f(b_r)$ is a pdf for all possible sums b_r in B.

S11 also can be modified to produce *pessimistic solutions* similar to S5-S7 above. In the case of S5 we will get a widest set of options to select values of variables, which the traveler can control from a variety of wake up times to different duration of breakfast and times to leave home under the assumption that uncontrolled variables will have their longest durations (pessimistic assumption).

2.3. Probabilistic Settings

Now consider the **probabilistic settings** where in Eq. (3) all X_i, D_k, and B are given with pdfs. As we have already stated Eq. (3) provides an incomplete mathematical formulation and needs to be augmented with specifications. We will continue to analyze the airport task for this case.

The probabilistic settings with different specifications produce different pdfs for B. At this moment we will concentrate not on these differences, but on types of tasks that can be formulated having a pdf. The most obvious one is a search for the solution that maximizes this pdf as was the case in S11 above.

What is the meaning of this solution in general – and, in particular, for the original airport task? To clarify this we need first to analyze the meaning of the pdfs for X_i and D_j. The traveler cannot influence and ignore pdfs for all D_j. Pdfs must be taken into account. The situation with X is different. Consider the breakfast duration, which is from X. For instance, this duration can have a unimodal pdf on interval from 10 to 20 minutes with the mean of 15 minutes. This pdf can be ignored and the traveler can use any time from this interval if needed.

Respectively maximization of probabilities can be done only using probabilities coming from all D_j. In this case the solution with max of pdf, max $f(b_r)$, will be the solution that uses most probable values of variables that are not controlled by the travelers such as duration of driving to the airport (due to traffic pattern). This solution has an obvious drawback—it does not guarantee him catching up with the plane. This strategy will work only statistically when a traveler is interested in catching multiple planes say over the year with the highest probability.

This is not the case in the task, when the traveler needs to catch for sure the specific plane at the given date. In this case he would need to use some solutions of the pessimistic task specification and max of pdf can be an additional requirement for the pessimistic task specification.

As a "pessimistic" solution, this solution does not use pdfs for D_i, but the worst cases of all D_i (longest durations). If also pdfs for X are ignored, then it becomes the interval-based formulation that we already discussed above. In both cases when pdfs on X are ignored or not ignored, the max of pdf on B will generate the largest number of alternative solutions that ensure that the traveler will not miss the plane.

2.4. Fuzzy sets settings

Consider the **fuzzy set settings** where in Eq. (3) all X_i, D_k, and B are given as fuzzy set/fuzzy numbers. The Zadeh's Extension Principle (EP) is doing exactly the same as max of pdf in the probabilistic setting, but for the membership functions of fuzzy sets.

It does not guarantee that the traveler will catch the plane. The catching the plane for sure requires a pessimistic definition of the solution. The extension principle provides solutions that maximize the membership function, $m(b_r)$ that represents sums

$x_1+x_2+x_3+x_4+d_1+d_2+\ldots+d_k=b_r$. The meaning of this maximization will depend on the definition of $m(b_r)$.

If $m(b_r)$ is a probability-based function (Kovalerchuk, 2014, 2015), then it is in essence the same as it is in the probabilistic setting discussed above. If it is in a possibilistic setting, then it is less clear because the possibilistic interpretation has no common operational definition, as this concept is defined by Bridgeman (1927) in his operationalism theory.

The optimistic solution in probabilistic and fuzzy sets settings is a solution with the highest probability or membership value, respectively. Similarly the pessimistic solution is a solution with the lowest probability and membership values, respectively. In these terms the *extension principle* is *optimistic*.

However in the probabilistic interpretation of the membership function, it may or may not be optimistic in the probabilistic sense. The reason is that it is an upper estimate of the probability, and it is not necessary that the upper estimate for the max of the probability will be higher than that for other alternatives. *Fuzzy extension principle is not an optimistic estimate for the probability*, but rather its upper estimate that may never be reached.

Now consider the EP for a pure fuzzy set interpretation of a membership function. EP uses the min operation which is an upper limit of all other t-norms,

$$m_t(A\&B) \leq min(A,B)$$

(Klement et al, 2000). If, in the task at hand, another t-norm m_a models that task more accurately, (e.g., *subjective comfort* of timing of all activities for catching the plane on time) then the min operation will provide only an upper estimate for the actual membership value, similarly to the upper estimate of the probability,

$$P(A\&B) \leq min(P(A,P(B)).$$

As a result, similarly to the probability case discussed above, a solution with the highest upper estimate of this m_a provided by the EP, not necessarily belongs to the alternative with the highest m_a value. Respectively the *fuzzy extension principle is not an optimistic estimate for another t-norm*, but rather it's the upper estimate that may never be reached.

This property allows using the EP to reject a solution, e.g., to reject a solution with the EP estimate equal to 0.3, when a desired membership value is 0.8. However, the EP solution with the 0.8 upper estimate, may not be good enough, because the actual value can be just 0.4.

The important point in this analysis is that the abstract sum of fuzzy sets $X_1+X_2+X_3+X_4$ -- defined without context of the airport task -- will be useless for solving this task. Unfortunately such practice still continues in some works. Next without specification a solution of Eq. (1) is not a solution of the catching the plane task. It only gives the constraint that needs to be accompanied by an optimization objective function, or a set of objective functions in a multi-objective optimization setting.

3. Scheduling as a task of solving a set of linear equations under uncertainty

3.1. General formulation

Below we consider the three linear equations under uncertainty. These equations provide the formulation of a scheduling task, which can be applied for designing an advanced automated personal scheduling assistance system:

$$X_1+A_{11}+A_{12}+A_{13}=B_1 \quad (4)$$

where X_1 is wake up time, A_{11} is washing time, A_{12} is time for eating breakfast, A_{13} is time to reach airport, B_1 is arrival time to the airport.

$$B_1+C_1+C_2=A_2 \quad (5)$$

where B_1 is the same arrival time to the airport as in (4), C_1 is duration of time at the airport before departure, C_2 is duration of flight time, and A_2 is landing time at the destination.

$$A_2+ A_{31} + A_{32} + A_{33} + X_2= B_2 \quad (6)$$

where A_2 is the same landing time at the destination as in (5), X_2 is time for lunch at the airport, A_{31} time to get luggage, A_{32} is time to get a rented car, (take shuttle, do renting paperwork), A_{33} is time to reach meeting place from the airport, and B_2 is the time of arrival to the meeting place.

The goal is solving these equations relative to X_1 and X_2, i.e., to find wake up time and duration of lunch time under uncertainty of all components of these equations given as intervals, pdfs, or fuzzy sets. Respectively, it is expected that X_1 and X_2 will be found as intervals, pdfs, or fuzzy sets. These equations have two other components A_{11} and A_{12} (washing and breakfast time), which a traveler can control more than others. These components can be included as variables that **need to be found, i.e., renamed as X_3 and X_4, respectively.**

3.2. Specifications

In the interval setting, equations (4)-(6) can be specified sequentially. The Eq (4) in essence is the same as Eq (2) with the only difference in notation. Respectively solution formulations S1-S11 all are applicable as formal definitions of the solutions to (4) and can be analyzed to be meaningful for the task. This analysis leaves only the pessimistic solution definitions S5-S7 for further analysis to ensure catching the plane and being on time at the meeting.

Assume that formulation of the solution S5 is picked up for (4). Then the solution definition for (5) can be the classical Moore's sum of intervals:

$$A_2=[A_{2s}, A_{2e}]= [B_{1s}+C_{1s}+C_{2s}, B_{1e}+C_{1e}+C_{2e}].$$

Similarly to (4) for (6) the options S1-S10 first are narrowed to S5-S7 and one of them is selected as the best fit for the task goal. For instance it can be S6 (the earliest pessimistic formulation). This example illustrates that having 3 uncertain equations for this task leads to three different definitions of the solution: S5, Moore's interval arithmetic, and S6. The standard approaches commonly assume a single definition for all equations.

This formulation of solution for (4)-(6) allows incorporating probabilistic and fuzzy sets formulations into it, because all pessimistic formulations are covered by (S5). Additional limitations can be imposed on a set of pessimistic solutions S5, e.g., requiring the use of solutions, which provide the probability or the membership value above some threshold or max of their values, respectively. The max case is equivalent to S11 and the former one is

$$\text{(S12)}\ \forall i\ \forall d_i \in D_i\ \forall\ b \in B\ \forall j\ \ \exists\ x_j \in X_j: \\ x_1+x_2+x_3+x_4+d_1+d_2+\ldots+d_k=b\ \&\ f(b_r)>T,$$

where T is a threshold.

Note that S6 is very rigid solution formulation without much room to incorporate additional limitations. Putting extra limitations on Eq. (5) is also questionable. Removing some possible landing times can hurt the possibility of solving the next equation (6).

4. Difference and differential equations

Consider a *difference equation under uncertainty*

$$\Delta x(t) = x(t + h) - x(t) = g(t, x(t)) \qquad (7)$$
$$x(t_0) = x_0$$

where $t \in [t_0, T]$ and g is a continuous function, and its equivalent form

$$x(t_n)=x(t_0)+\Delta x(1)+ \Delta x(2)+\ldots +\Delta x(n-1), \qquad (8)$$

(Allahviranloo et al, 2016). Here, in contrast with the crisp case, all components of this equation are uncertain, i.e., intervals, pdfs, or fuzzy sets. Thus, in the interval setting, all x(t) are intervals, respectively $\Delta x(t)=x(t+h)-x(t)=g(t,x(t))$ are also intervals. In the probabilistic setting we need to sum up or subtract two or more pdfs by computing the joint pdfs, which is commonly an ill-posed problem due to lack of information on relations between the pdfs in the form of conditional probabilities. In the fuzzy sets setting, the common way to define the difference gH of two fuzzy numbers x and y, is the generalized Hukuhara difference (Stefanini, 2010),

$$x -_{gH} y = w \Leftrightarrow x = y + w.$$

Respectively the sum and difference of intervals, pdfs, or fuzzy sets must be defined in a meaningful way similarly to what was proposed above for linear equations in S1-S12 specifications.

Example. Consider a car moving on the road/highway for an hour. The task is predicting the travel distance $x(t)$ at t=3600 seconds, i.e., the location of the car on the road in one hour using (8) in the interval setting. Assume that for every second t the interval formula $x(t)=x(t-1)+\Delta x(t)$ computes the distance $x(t)$ with the accuracy ±3m, i.e., as the interval

$$[x(t-1)+\Delta x(t)-3,\ x(t-1)+\Delta x(t)+3].$$

Next for simplicity assume that $\Delta x(t)= \Delta x(1)$ for all t. Then, the use of Moore's interval arithmetic gives the total travel distance interval

$$[x(0)+3600\Delta x(1)-10800,\ x(0)+ 3600\Delta x(1)+10800].$$

Thus, the error interval grows 3600 times to ±10800m, [-10800,10800] from [-3,3], i.e., prediction error interval is 21.6 km, which is 21.6% error for the average speed of 100km/h. This interval definition of the solution does not produce any more specific answer, such as a most likely/probable travel distance within this large error interval.

In contrast, the probabilistic definition of solution under assumption of uniform and independent distribution of errors in each [-3,3] interval for each t leads to the solution convolution formula for the total probability (Kovalerchuk, 2014). The max of that probability is reached at the middle of the travel distance $x(0)+3600\Delta x(1)$.

The change of the uniform error distribution on the interval [-3,3] to a unimodal pdf (e.g., normal distribution) will keep the max of the total pdf of the sum in the middle of the distance interval $x(0)+3600\Delta x(1)$.

The change of error pdfs to fuzzy numbers in the [-3,3] centered at 0 will keep the max of the total membership of the sum in the middle of the distance interval $x(0)+3600\Delta x(1)$. This is the EP solution for difference equations. Here EP is conceptually doing the same *maximization of the upper estimate of the membership* as it does for linear equations discussed above.

Thus, the probabilistic and fuzzy sets solutions allow to *deeper characterize* the points in the prediction interval, but still do not allow selecting any of them. The further specification of the task allows this, as we show below using two specifications of the prediction task.

Task 1 is using the prediction for telling the taxi driver, where and when to meet this car on the road for giving this passenger a ride to the airport from that meeting point. To do this for sure, the definition of the solution will require to use the pessimistic solution definition - starting point of the prediction interval, which, in this example, is $x(0)+3600\Delta x(1)-10800$.

Task 2 is identifying the location B on the road, for building a rest area, which should be within one hour of driving from point A. This location B will be an interval with the probabilities $p(x)$, or the membership values $m(x)$ for the points in B. The reasonable definition of the solution here will be a point with the max of that probability or membership value. As we see, task 2 requires a very different definition of the solution than task 1.

Differential equations under uncertainty with intervals, pdfs and fuzzy sets *generalize* the difference equation (7) to the continuous setting,

$$dx(t)/dt = g(t, x(t)) \quad (9)$$

$$x(t_0) = x_0$$

Often differential equations are solved numerically by converting to the difference equations. Therefore all different formulations of solutions for the difference equations presented above are *applicable* to the differential equations too.

5. LESSONS FROM PRIOR STUDIES

5.1. Defining solution for fuzzy differential equations

As was stated in the introduction, most of the activities in the fuzzy systems for fuzzy equations were concentrated on finding solutions and conditions, when solutions exist under the definition of solution based on the Zadeh's extension principle. The paper by Buckley and Feuring (2000) is a representative sample of this type of studies for both algebraic and differential equations. They start from the first-order ordinary differential equation

$$dy/dt = f(t,y,k), \ y(0) = c, \quad (10)$$

where $k = (k_1, ..., k_n)$ is a vector of constants, and t is in some closed and bounded interval, which contains zero. It is assumed that f satisfies some conditions so that Eq. (10) has an *unique crisp solution* $y= g(t,k,c)$, for $t\in l$, $k\in K\subset R^n$, $c\in C\subset R$.

The fuzzification is defined by introduction of a vector of triangular fuzzy numbers $K^\wedge = (K^\wedge_i, ..., K^\wedge_n)$ and another triangular fuzzy number $C^\wedge$. Then k is substituted by $K^\wedge$ and c is substituted by $C^\wedge$ in (10) to get

$$dY^\wedge/dt = f(t,Y^\wedge,K^\wedge), \quad f(0) = C^\wedge \quad (11)$$

under the assumption of some definition for the derivative of the fuzzy function *f(t)* from the literature and that f is a fuzzy number for each t. Solving (11) is defined as finding $Y^\wedge(t)$ that itself is defined in three *equivalent ways*:

(W1) by fuzzification of the crisp solution $y=g(t,k,c)$ using the *extension principle* to get $Y^\wedge(t)=g(t,K^\wedge,C^\wedge)$

(W2) by using *α-cuts* $K(\alpha) = K^\wedge_1[\alpha] \times \cdot\cdot\cdot \times K^\wedge_n[\alpha]$ and $\Phi(\alpha)=K(\alpha) \times C[\alpha]$, for $0\leq \alpha\leq 1$ to get *α*-cuts

$$Y(t)[\alpha] = [Y_1(t,\alpha), Y_2(t,\alpha)],$$

where

$$Y_1(t,a)= min\{g(t, k,c) \,|k \in K^\wedge[a],\, c \in C[a]\} \text{ and}$$
$$y_2(t,a) = max\{g(t,k,c) \,|k \in K^\wedge[a],\, c \in C[a]\},$$

(W3) by using *α-cuts* $\Omega(\alpha) = \{g(t,k,c) \mid (k, c) \in \Phi(\alpha)\}$ for $0\leq \alpha\leq 1$, and $t \in I$ as the membership function

$$Y^\wedge(t)(x) = sup\{ \alpha \mid x\in \Omega(\alpha)\}$$

These authors gave necessary and sufficient conditions for $Y(t)$ to solve this fuzzy initial value problem. The extension to fuzzy partial differential equations is proposed in the same way as finding the crisp solution, fuzzifying it, and then checking to see if it satisfies the fuzzy partial differential equation. As we see the extension principle is accepted without discussing the justification of definition of the solution based on the extension principle.

5.2. Systems of stochastic linear algebraic equations

Girko in a series of publications (1992, 1996, 1998) discussed solutions for a *system of linear algebraic equations* (SLAE) $Ay=b$, when A and b are given with some random errors. He pointed out (Girko, 1992) that "it is not yet clear how to find the best *consistent*, in some sense, estimates of the solutions of SLAE, if their coefficients are given with some random errors, and conditions of existence of moments of the components y_k of the vector y have not been found." This author provides an equation (12) presented below, where A is a matrix of order n × m and a transposed matrix is denoted by a "prime". The expression (12) is called a *regularized pseudo-solution* of SLAE $Ay=b$ with the random coefficients and nonsingular matrices C_1, C_2 that provide regularization, where $\alpha\geq 0$, $\beta>0$ are some constants,

$$y_a = (C_1'C_1\alpha + A'(C_2')^{-1}C_2^{-1}A\beta^{-1})^{-1}A'(C_2')^{-1}C_2^{-1}\beta^{-1}b \quad (12)$$

In this work the main focus was on the mathematical issues of funding approximation of solution for A with a large n. Our focus is on the question of the justification of a definition of the solution. The regularized solution needs to be justified for each applied task that includes the justification of α, β, C_1, and C_2 in (12). This question was left unanswered in that work.

Provencher (1982) pointed out that the problems of stochastic linear equations generally have a large number of possible solutions (the *ill-posed inversion problem*) with *arbitrary large deviations* from each other all of which fit to the error distribution functions obtained experimentally. Kac (1943) studied the average number of real roots of a specific random algebraic equation $X_0+X_1x+X_2x^2+\ldots+X_{n-1}x^{n-1}=0$, where the X's are independent random variables with the same normal distribution. As we see it is not a general linear stochastic equation, but a quite specific one that allowed estimating the average number of solutions.

Respectively Provencher (1982) stated "straightforward inversion procedures cannot be used and statistical *regularization techniques are necessary*". Then he discusses the most relevant for us, the issue of selecting and justifying the regularization method. The idea is to find a *simplest solution* that is consistent with prior knowledge and experimental data that can be available in addition to the stochastic equation. While simplicity can be achieved and was achieved by multiple methods including listed in (Provencher, 1982), its *relevance* to the task at hand is not obvious. In the common approach, the regularizer will impose *simplicity* (typically *smoothness*) or *statistical prior knowledge*.

The search for such simplest regularizers can be conducted by solving a quadratic optimization problem, and by using F-test and confidence regions. Methods of explicit solving of stochastic systems of linear algebraic equations, which include the Monte Carlo method, the perturbation method, the Neumann expansion method and the polynomial chaos have been reviewed in (Li et al, 2006). In essence simplicity is an *external criterion* to the task at hand. For our *airline task* experimental data on the duration of breakfast and driving to the airport will can shift the solution to the averages of these values, not to the pessimistic durations that will guarantee that the plane will not be missed.

5.3. Stochastic programming

As we see the regularization actually converts solving the stochastic equation/equations into solving an optimization task where the original equation/equations can be modified or used as constraints. The advantage of the explicitly stated optimization task is that it allows to clearly *separating the technical external-to-the-task assumptions from the ones derived from the task and relevant to the domain knowledge*. The same is applicable for solving fuzzy equations by solving the fuzzy optimization tasks.

The assumptions of the fuzzy optimization tasks have been reviewed in (Kovalerchuk, 1994) and seem still valid. For instance, in the classical two-stage linear stochastic programming problems (Shapiro et al, 2009, King, Wallace, 2012) it is clearly stated that, at the first stage, we minimize the cost of the first-stage decision plus the *expected cost* of the second-stage decision. It is assumed that the second-stage cost is a random vector with a *known* probability distribution. It means that we deal with the randomness of the second-stage cost by minimizing its average (expected cost). If, for a particular task, this is not appropriate, like in our airline task, the objective function must be rewritten appropriately.

6. Conclusion

In many practical situations, we need to solve equations and systems of equations under uncertainty. This uncertainty can be interval, probabilistic, fuzzy, etc. To solve such problems, at first glance, it seems reasonable to take the solution to the corresponding exact systems, and apply the general translation to the corresponding type of uncertainty – interval computations for interval uncertainty, the extension principle for fuzzy uncertainty, etc. This is exactly how many researchers and practitioners often solve the corresponding uncertain problems. In this paper, we emphasize that the resulting solutions are sometimes inadequate. The reason for this inadequacy is that to come up with a correct solution, we need to analyze the original problem under uncertainty, not only its exact prototype. Often, for exact data, different practical problems lead to the same solution, while *in the presence of uncertainty* these problems often lead to *completely different solutions* as was shown in this paper. This ambiguity cannot be resolved by simply modifying the usual formal approach: e.g., even for the simple case, when the exact-case solution is just a subtraction, in the uncertainty case, there can be at least ten different definitions of a solution. Our recommendation is to always take into account the meaning of the corresponding real-world problem as opposed to just the equations. This is in perfect accordance with the original spirit of fuzzy systems approach: to properly analyze real-life systems, we need to take into account not only the corresponding equations, but also the experts' knowledge that goes beyond these equations.

Accordingly the contribution of this paper is the seven new formalized definitions of the concepts of solutions for the linear equations with uncertain components that are applicable to interval, probability and fuzzy set settings and are interpretable in the real world tasks. The clearly interpretable concept of optimistic and pessimistic solutions enabled the conversion of ill-posed problems into complete mathematical tasks using these new definitions. It is shown that these definitions have generalizations for difference and differential equations under uncertainty. `

The multiplicity of the definitions of solutions of equations under uncertainty creates an exciting opportunity for new mathematical and algorithmic research on the finding of efficient methods to solve the equations under these new definitions.

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