A classification of representable t-norm operators for picture fuzzy sets

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***Abstract*—T-norms and t-conorms are basic operators of fuzzy logics. The classifications of these operators are significant problems. Some results of the classifications of fuzzy logics operators for fuzzy sets are given in [3,4]. In 2013, we defined the picture fuzzy sets [5,6] and in 2015 some representable t-norms operators and t-conorms operators were defined and firstly were presented in [7,8]. In this paper, we will investigate the classification of representable picture t-norms and picture t-conorms operators for picture fuzzy sets.**

I. Introduction

Since Zadeh have motivated the fuzzy sets (FSs) in 1965 [1], many theories treating imprecition and uncertainly have been introduced. Some of these are extentions of fuzzy set theory ([3], [4] - [11], [12]). In the 1980s, Atanassov proposed an generalization of fuzzy sets, fuzzy logic - intuitionistic fuzzy sets, intuitionistic fuzzy logic [10]. While the theory of fuzzy sets based approximate reasoning is well-established, there is still a demand for the expressiveness of the formalism. In 2013, a new concept - "picture fuzzy sets" - a direct generalization of Zadeh' fuzzy sets and Atanassov' intuitionistic fuzzy sets were introduced. Then new operators of picture fuzzy sets were presented with many properties were given in [5,6]. In [7], we have introduced picture fuzzy logic connectives and some properties of picture fuzzy sets. In this paper, we will study some classifications of representable picture fuzzy t-norms and picture fuzzy t-conorms on picture fuzzy sets.

Definition 1.1. [11]. An intuitionistic fuzzy set (IFS) A on the universe X is an object of the form

$$A = \{(x, \mu_A(x), \nu_A(x))|x \in X\},$$

where $\mu_A(x) \in [0, 1]$ is called the "degree of membership of x in A", $\nu_A(x) \in [0, 1]$ is called the "degree of non-membership of x in A", and $\mu_A(x)$ and $\nu_A(x)$ satisfy

$$\mu_A(x) + \nu_A(x) \leq 1, \forall x \in X.$$

In this paper, let $IFS(X)$ denote the set of all intuitionistic fuzzy sets on X.

A generalization of fuzzy sets and intuitionistic fuzzy sets is the following notion of picture fuzzy sets.

Definition 1.2. [5,6]. A picture fuzzy set A on the universe X is an object of the form

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x))|x \in X\},$$

where $\mu_A(x) \in [0, 1]$ is called the "degree of positive membership of x in A", $\eta_A(x) \in [0, 1]$ is called the "degree of neutral membership of x in A", and $\nu_A(x) \in [0, 1]$ is called the "degree of negative membership of x in A", and $\mu_A(x), \eta_A(x), \nu_A(x)$ satisfy

$$\mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1, \forall x \in X.$$

Then, $\forall x \in X, 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x))$ is called the "degree of refusal membership of x in A".

Basically, picture fuzzy sets based models may be adequate in situations when decision makers face their opinions involving their decision making as follows: yes, abstain, no, and refusal. The voting results are divided into four groups accompanied with the number of voters that are "voters for", "abstain", "vote against", and "refusal of the voting".

In this paper, following the research in [5,6], we investigate basic picture fuzzy logic connectives: negation, conjunction, disjuction. A definition and some properties of picture fuzzy negators, picture fuzzy t-norms, picture fuzzy t-conorms are represented in Section 2. Section 3 is drawn for some properties of picture t-norms and picture t-conorms. In Section 4, some subclasses of representable picture t-norms for a classification are presented. A similar classification for representable picture t-conorms is given in Section 5. The paper closes with Section 6, where we set out concluding fuzzy inference processes in picture fuzzy systems with consideration of potential solutions, open research questions for research.

Let $PFS(X)$ denote the set of all the picture fuzzy sets on the universe X.

Now, we consider the set D^* defined by

$$D^* = \{x = (x_1, x_2, x_3)|x \in [0, 1]^3, x_1 + x_2 + x_3 \leq 1\}.$$

From now on, we will assume that if $x \in D^*$, then x_1, x_2 and x_3 denote, respectively, the first, the second and the third component of x, i.e., $x = (x_1, x_2, x_3)$.

Consider the order relation $\leq_1$ on D^*, defined by $x \leq_1 y \Leftrightarrow ((x_1 < y_1) \wedge (x_3 \geq y_3)) \vee ((x_1 = y_1) \wedge (x_3 > y_3)) \vee ((x_1 =$

$y_1) \wedge (x_3 = y_3) \wedge (x_2 \leq y_2))$, for all $x \in D^*$.

We define the first, second and third projection mapping pr_1 and pr_2 and pr_3 on D^*, defined as $pr_1(x) = x_1$ and $pr_2(x) = x_2$ and $pr_3(x) = x_3$, for all $x \in D^*$.

For each $x, y \in D^*$, we define

$$\inf(x,y) = \begin{cases} \min(x,y), & \text{if } x \leq_1 y \text{ or } y \leq_1 x \\ (x_1 \wedge y_1, 1 - x_1 \wedge y_1 - x_3 \vee y_3, x_3 \vee y_3), & \text{else} \end{cases}$$

$$\sup(x,y) = \begin{cases} \max(x,y), & \text{if } x \leq_1 y \text{ or } y \leq_1 x \\ (x_1 \vee y_1, 0, x_3 \wedge y_3), & \text{else.} \end{cases}$$

Then, $(D^*, \leq_1)$ is a complete lattice [3]. Indeed, for each nonempty $A \subseteq D^*$, we have

$$\inf A = (\inf pr_1 A, \inf pr_2 A, \inf pr_3 A), \text{where}$$

$\inf pr_1 A = \inf \{x_1 \in [0,1] \,|\exists x = (x_1, y_2, x_3) \in A\}$,
$\inf pr_3 A = \sup\{x_3 \in [0,1] \mid \exists\, x = (x_1, x_2, x_3) \in A\}$,

$$\inf pr_2 A = \begin{cases} \inf\{x_2 : (\inf pr_1 A, x_2, \inf pr_3 A) \in A\} \\ 1 - \inf pr_1 A - \inf pr_3 A, \\ \text{if}(\inf pr_1 A, z, \inf pr_3 A) \notin A,\ \forall z \end{cases} \quad \text{and}$$

$$\sup A = (\sup pr_1 A, \sup pr_2 A, \sup pr_3 A), \text{where}$$

$\sup pr_1 A = \sup\{x_1 \in [0,1] \mid \exists\, x = (x_1, x_2, x_3) \in A\}$,
$\sup pr_3 A = \inf\{x_3 \in [0,1] \mid \exists\, x = (x_1, x_2, x_3) \in A\}$,

$$\sup pr_2 A = \begin{cases} \sup\{x_2 : (\sup pr_1 A, x_2, \sup pr_3 A) \in A\}, \\ 0, \quad \text{if}(\sup pr_1 A, z, \sup pr_3 A) \notin A, \forall z. \end{cases}$$

We denote the units of D^* by $1_{D^*} = (1,0,0)$ and $0_{D^*} = (0,0,1)$, respectively.

Note that, if for $x, y \in D^*$ that neither $x \leq_1 y$ nor $y \leq_1 x$, then x and y are incomparable w.r.t. $\leq_1$, denoted as $x||_{\leq_1} y$.

Using this lattice, we easily see that with every picture fuzzy set $A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) | \, x \in X\}$ corresponds an D^*-fuzzy set [10], i.e., a mapping $A : X \to D^* : x \mapsto (\mu_A(x), \eta_A(x), \nu_A(x))$.

II. Picture fuzzy t-norms and Picture fuzzy t-conorms

Picture fuzzy negators form an extension of fuzzy negators [3] and intuitionistic fuzzy negators [14], and are defined as follows.

Definition 2.1. A picture fuzzy negator is any nonincreasing $D^* \to D^*$ mapping N satisfying $N(0_{D^*}) = 1_{D^*}$, and $N(1_{D^*}) = 0_{D^*}$. If $N(N(x)) = x$, for all $x \in D^*$, then N is called an involutive negator.

The mapping N_0 defined by $N_0(x) = (x_3, 0, x_1)$, for all $x \in D^*$, be a picture fuzzy negator.

From now on, if $x \in D^*$, then $x_4 = 1 - x_1 - x_2 - x_3$.

The mapping N_S defined by $N_S(x) = (x_3, x_4, x_1)$, for all $x \in D^*$, will be called the *standard* negator.

Many properties of the involutive picture fuzzy negators were given in [8].

Now we define picture fuzzy t-norms and picture fuzzy t-conorms and will give some classes of conjuction operators and some classes of disjunction operators for picture fuzzy sets. Picture fuzzy t-norms are direct extension of fuzzy t-norms (see [3, 16, 17]) and of intuitionistic fuzzy t-norms [14].

Let for each x, we denote

$$I(x) = \{y \in D^* : y = (x_1, y_2, x_3), 0 \leq y_2 \leq 1 - (x_1 + x_3)\}$$

Definition 2.2. A picture fuzzy t-norm is an $(D^*)^2 \to D^*$ mapping T satisfying the folowing conditions:

* $T(x,y) = T(y,x), \forall x, y \in D^*$ (commutativity),
* $T(x, T(y,z)) = T(T(x,y), z), \forall x, y, z \in D^*$ (associativity),
* $T(x,y) \leq_1 T(x,z), \forall x, y, z \in D^*, y \leq_1 z$ (monotonicity), and
* $T(1_{D^*}, x) \in I(x), \forall x \in D^*$ (boundary condition).

Definition 2.3. A picture fuzzy t-conorm is an $(D^*)^2 \to D^*$ mapping S satisfying the folowing conditions:

* $S(x,y) = S(y,x), \forall x, y \in D^*$ (commutativity),
* $S(x, S(y,z)) = S(S(x,y), z), \forall x, y, z \in D^*$ (associativity),
* $S(x,y) \leq_1 S(x,z), \forall x, y, z \in D^*, y \leq_1 z$ (monotonicity), and
* $S(0_{D^*}, x) \in I(x), \quad \forall x \in D^*$ (boundary condition).

From now on, we denote $x \wedge y := \min(x,y), x \vee y := \max(x,y)$ for all $x, y \in [0,1]$.

Example 2.4. Some picture fuzzy t-norms, for all $x, y \in D^*$:

* $T_{\inf}(x,y) = \inf\{x,y\} = \begin{cases} (x_1 \wedge y_1, 1 - (x_1 \wedge y_1) - (x_3 \vee y_3), x_3 \vee y_3), & x||_{\leq_1} y \\ x \wedge y, & \text{otherwise.} \end{cases}$
* $T_{\min}(x,y) = (x_1 \wedge y_1, x_2 \wedge y_2, x_3 \vee y_3)$.
* $T(x,y) = (x_1 \wedge y_1, x_2 y_2, x_3 \vee y_3)$.
* $T(x,y) = (x_1 y_1, x_2 y_2, x_3 \vee y_3)$.
* $T(x,y) = (x_1 y_1, x_2 y_2, x_3 + y_3 - x_3 y_3)$.
* $T(x,y) = \left(\begin{cases} x_1 \wedge y_1 \text{ if } x_1 \vee y_1 = 1 \\ 0 \text{ if } x_1 \vee y_1 < 1 \end{cases}, \begin{cases} x_2 \wedge y_2 \text{ if } x_2 \vee y_2 = 1 \\ 0 \text{ if } x_2 \vee y_2 < 1 \end{cases}, \begin{cases} x_3 \vee y_3 \text{ if } x_3 \wedge y_3 = 0 \\ 1 \text{ if } x_3 \wedge y_3 \neq 0 \end{cases}\right)$.
* $T(x,y) = (0 \vee (x_1 + y_1 - 1), 0 \vee (x_2 + y_2 - 1), 1 \wedge (x_3 + y_3))$.
* $T(x,y) = (0 \vee (x_1 + y_1 - 1), 0 \vee (x_2 + y_2 - 1), x_3 + y_3 - x_3 y_3)$.
* $T(x,y) = (\frac{1}{2}(x_1 + y_1 - 1 + x_1 y_1) \vee 0, \frac{1}{2}(x_2 + y_2 - 1 + x_2 y_2) \vee 0, x_3 + y_3 - x_3 y_3)$.
* $T(x,y) = (x_1 y_1, 0 \vee (x_2 + y_2 - 1), x_3 + y_3 - x_3 y_3)$.
* $T(x,y) = (0 \vee (x_1 + y_1 - 1), x_2 y_2, x_3 + y_3 - x_3 y_3)$.

Example 2.5. Some picture fuzzy t-conorms, for all $x, y \in D^*$:

* $S_{\sup}(x,y) = \sup\{x,y\} = \begin{cases} (x_1 \vee y_1, 0, x_3 \wedge y_3), & \text{if } x||_{\leq_1} y \\ x \vee y, & \text{otherwise.} \end{cases}$
* $S_{\max}(x,y) = (x_1 \vee y_1, x_2 \wedge y_2, x_3 \wedge y_3)$.
* $S(x,y) = (x_1 \vee y_1, x_2 y_2, x_3 \wedge y_3)$.
* $S(x,y) = (x_1 \vee y_1, x_2 y_2, x_3 y_3)$.
* $S(x,y) = (x_1 + y_1 - x_1 y_1, x_2 y_2, x_3 y_3)$.

* $S(x,y) = (x_1 \vee y_1, \begin{cases} x_2 \wedge y_2 \text{ if } x_2 \vee y_2 = 1 \\ 0 \text{ if } x_2 \vee y_2 < 1 \end{cases}, x_3 \wedge y_3)$.
* $S(x,y) = (1 \wedge (x_1 + y_1), 0 \vee (x_2 + y_2 - 1), 0 \vee (x_3 + y_3 - 1))$.
* $S(x,y) = (x_1 + y_1 - x_1 y_1, 0 \vee (x_2 + y_2 - 1), 0 \vee (x_3 + y_3 - 1))$.
* $S(x,y) = (x_1 + y_1 - x_1 y_1, \frac{1}{2}(x_2 + y_2 - 1 + x_2 y_2) \vee 0, \frac{1}{2}(x_3 + y_3 - 1 + x_3 y_3) \vee 0)$.
* $S(x,y) = (x_1 + y_1 - x_1 y_1, 0 \vee (x_2 + y_2 - 1), x_3 y_3)$.
* $S(x,y) = (x_1 + y_1 - x_1 y_1, x_2 y_2, 0 \vee (x_3 + y_3 - 1))$.

In 2004, G.Deschrijver, C.Cornelis and E.E.Kerre [12] introduced the notion of intuitionistic fuzzy t-norms and t-conorms and investigated under which conditions a similar representation theorem could be obtained.

For further usage, we define $L^* = \{x \in D^* | x_2 = 0\}$.

Now, we can consider the set L^* defined by

$$L^* = \{u = (u_1, u_3) | u \in [0,1]^2, u_1 + u_3 \leq 1\}.$$

Consider the order relation $u \leq v$ on L^*, defined by $u \leq v \Leftrightarrow ((u_1 \leq v_1) \wedge (u_3 \geq v_3))$, for all $u \in L^*$.

We define the first, and second projection mapping pr_1 and pr_3 on L^*, defined as $pr_1(u) = u_1$ and $pr_3(u) = u_3$, for all $u \in L^*$.

We denote the units of L^* by $1_{L^*} = (1,0)$ and $0_{L^*} = (0,1)$.

Definition 2.6. [12]. An intuitionistic fuzzy t-norm is a commutative, associative, increasing $(L^*)^2 \to L^*$ mapping T satisfying $T(1_{L^*}, u) = u$, for all $u \in L^*$.

Definition 2.7.[12]. An intuitionistic fuzzy t-conorm is a commutative, associative, increasing $(L^*)^2 \to L^*$ mapping S satisfying $S(v, 0_{L^*}) = v$, for all $v \in L^*$.

Definition 2.8. An intuitionistic fuzzy t-norm T is called t-representable iff there exist t-norm t_1, and a t-conorm s_3 on [0,1] satisfying, for all $u, v \in L^*$,

$$T(u,v) = (t_1(u_1, v_1), s_3(u_3, v_3)).$$

Definition 2.9. An intuitionistic fuzzy t-conorm S is called t-representable iff there exist t-norm t_1, and a t-conorm s_3 on [0,1] satisfying, for all $u, v \in L^*$,

$$S(u,v) = (s_3(u_1, v_1), t_1(u_3, v_3)).$$

Definition 2.10. A picture fuzzy t-norm T is called representable iff there exist two t-norms t_1, t_2 and a t-conorm s_3 on [0,1] satisfying, for all $x, y \in D^*$,

$$T(x,y) = (t_1(x_1, y_1), t_2(x_2, y_2), s_3(x_3, y_3)).$$

Definition 2.11. A picture fuzzy t-conorm S is called representable iff there exist two t-norms t_1, t_2 and a t-conorm s_3 on [0,1] satisfying, for all $x, y \in D^*$,

$$S(x,y) = (s_3(x_1, y_1), t_2(x_2, y_2), t_1(x_3, y_3)).$$

The following propositions should give the new classes of picture t-norms from the intuionistic fuzzy t-noms. They present one way to construct the new picture fuzzy t-norms and t-representable picture fuzzy t-norms.

Proposition 2.12. Assume that $\mathcal{T}(u,v)$ is a t-representable intuitionistic fuzzy t-norm $\mathcal{T}(u,v) = (t_1(u_1, v_1), s_3(u_3, v_3))$, for all $u = (u_1, u_3), v = (v_1, v_3) \in L^*$, where t_1 is a t-norm on [0,1], s_3 is a t-conorm on [0,1]. Then $T(x,y) = (t_1(x_1, y_1), 0, s_3(x_3, y_3))$, for all $x, y \in D^*$ is a picture fuzzy t-norm.

Proposition 2.13. Assume that $\mathcal{S}(u,v)$ is a t-representable intuitionistic fuzzy t-conorm $\mathcal{S}(u,v) = (s_3(u_1, v_1), t_1(u_3, v_3))$, for all $u = (u_1, u_3), v = (v_1, v_3) \in L^*$, where t_1 is a t-norm on [0,1], s_3 is a t-conorm on [0,1]. Then $S(x,y) = (s_3(x_1, y_1), 0, t_1(x_3, y_3))$, for all $x, y \in D^*$ is a picture fuzzy t-conorm.

Proposition 2.14. Assume that $\mathcal{T}(u,v)$ is a t-representable intuitionistic fuzzy t-norm $\mathcal{T}(u,v) = (t_1(u_1, v_1), s_3(u_3, v_3))$, for all $u = (u_1, u_3), v = (v_1, v_3) \in L^*$, where t_1 is a t-norm on [0,1], s_3 is a t-conorm on [0,1]. Assume that t_2 is a t-norm on [0,1] satisfying, for all $x, y \in D^*$,

$$0 \leq t_1(x_1, y_1) + t_2(x_2, y_2) + s_3(x_3, y_3) \leq 1,$$

then $T(x,y) = (t_1(x_1, y_1), t_2(x_2, y_2), s_3(x_3, y_3))$, for all $x, y \in D^*$ is a representable picture fuzzy t-norm.

Proposition 2.15. Assume that $\mathcal{S}(u,v)$ is a t-representable intuitionistic fuzzy t-conorm $\mathcal{S}(u,v) = (s_3(u_1, v_1), t_1(u_3, v_3))$, for all $u = (u_1, u_3), v = (v_1, v_3) \in L^*$, where t_1 is a t-norm on [0,1], s_3 is a t-conorm on [0,1]. Assume that t_2 is a t-norm on [0,1] satisfying, for all $x, y \in D^*$,

$$0 \leq t_1(x_1, y_1) + t_2(x_2, y_2) + s_3(x_3, y_3) \leq 1,$$

then $S(x,y) = (s_3(x_1, y_1), t_2(x_2, y_2), t_1(x_3, y_3))$, for all $x, y \in D^*$ is a representable picture fuzzy t-conorm.

III. Some properties of picture fuzzy t-norms and picture fuzzy t-conorms

Now we give some properties of picture fuzzy t-norms and picture fuzzy t-conorms.

Definition 3.1. A picture fuzzy t-norm is called Achimerdean iff: $\forall x \in D^* \setminus \{0_{D^*}, 1_{D^*}\}, T(x,x) <_1 x$.

Definition 3.2. A picture fuzzy t-norm is called nilpotent iff: $\exists x, y \in D^* \setminus \{0_{D^*}\}, T(x,y) = 0_{D^*}$.

Definition 3.3. A picture fuzzy t-norm is called strict iff: $\forall x, y \in D^* \setminus \{0_{D^*}\}, T(x,y) \neq 0_{D^*}$.

Definition 3.4. A picture fuzzy t-conorm is called Achimerdean iff: $\forall x \in D^* \setminus \{0_{D^*}, 1_{D^*}\}, S(x,x) >_1 x$.

Definition 3.5. A picture fuzzy t-conorm is called nilpotent iff: $\exists x, y \in D^* \setminus \{1_{D^*}\}, S(x,y) = 1_{D^*}$.

Definition 3.6. A picture fuzzy t-conorm is called strict iff: $\forall x, y \in D^* \setminus \{1_{D^*}\}, S(x,y) \neq 1_{D^*}$.

Proposition 3.1. Let $T^* = \{\text{nilpotent picture t-norms}\}$ and $T^{**} = \{\text{strict picture t-norms}\}$. Then

$$T^* \cap T^{**} = \emptyset.$$

Proposition 3.2. Let $S^* = \{\text{nilpotent picture t-conorms}\}$ and $S^{**} = \{\text{strict picture t-conorms}\}$. Then

$$S^* \cap S^{**} = \emptyset.$$

Proposition 3.3. Let T be a representable picture fuzzy t-norm:

$$T\left(x,y\right)=\left(t_1\left(x_1,y_1\right),t_2\left(x_2,y_2\right),s_3\left(x_3,y_3\right)\right),\forall x,y\in D^*,$$

where t_1,t_2,s_3 are Archimedean [8], then T is Archimedean.

Proposition 3.4. Let S be a representable picture fuzzy t-conorm:

$$S\left(x,y\right)=\left(s_3\left(x_1,y_1\right),t_2\left(x_2,y_2\right),t_1\left(x_3,y_3\right)\right),\forall x,y\in D^*,$$

where t_1,t_2,s_3 are Archimedean [8], then S is Archimedean.

IV. A CLASSIFICATION OF REPRESENTABLE PICTURE T-NORMS

We have some the subclasses of representable picture fuzzy t-norms as follows:

1. Strict-strict-strict subclass of t-norms, denoted by Δ_{sss}:

Definition 4.1. A picture fuzzy t-norm T is called strict-strict-strict iff:

$$T\left(x,y\right)=\left(t_1\left(x_1,y_1\right),t_2\left(x_2,y_2\right),s_3\left(x_3,y_3\right)\right),\forall x,y\in D^*,$$

where t_1,t_2 are strict t-norms and s_3 is a strict t-conorm on [0,1].

Example 4.1.

* $T_1(x,y)=(x_1y_1,x_2y_2,x_3+y_3-x_3y_3)$,
* $T_2(x,y)=(\frac{x_1y_1}{\lambda_1+(1-\lambda_1)(x_1+y_1-x_1y_1)},\frac{x_2y_2}{\lambda_2+(1-\lambda_2)(x_2+y_2-x_2y_2)},(x_3^a+y_3^a-x_3^ay_3^a)^{\frac{1}{a}}),\lambda_1,\lambda_2,a\in[1,+\infty)$,

2. Nilpotent-nilpotent-nilpotent subclass of t-norms, denoted by Δ_{nnn}:

Definition 4.2. A picture fuzzy t-norm T is called nilpotent-nilpotent-nilpotent iff:

$$T\left(x,y\right)=\left(t_1\left(x_1,y_1\right),t_2\left(x_2,y_2\right),s_3\left(x_3,y_3\right)\right),\forall x,y\in D^*,$$

where t_1,t_2 are nilpotent t-norms and s_3 is a nilpotent t-conorm on [0,1].

Example 4.2.

* $T_3(x,y)=(0\vee(x_1+y_1-1),0\vee(x_2+y_2-1),1\wedge(x_3+y_3))$,
* $T_4(x,y)=(((x_1+y_1-1)(1+\lambda_1)-\lambda_1x_1y_1)\vee0,((x_2+y_2-1)(1+\lambda_2)-\lambda_2x_2y_2)\vee0,1\wedge(x_3^a+y_3^a)^{\frac{1}{a}}),\lambda_1,\lambda_2\in[0,+\infty),a\geq1$,
* $T_5(x,y)=((0\vee(x_1^a+y_1^a-1))^{\frac{1}{a}},(0\vee(x_2^b+y_2^b-1))^{\frac{1}{b}},1\wedge(x_3^c+y_3^c)^{\frac{1}{c}}),a,b,c\geq1$,
* $T_6(x,y)=((\frac{1}{a}(x_1+y_1-1+(a-1)x_1y_1)\vee0),(\frac{1}{b}(x_2+y_2-1+(b-1)x_2y_2)\vee0),1\wedge(x_3^c+y_3^c)^{\frac{1}{c}}),a,b\in(0,1];c\geq1$,
* $T_7(x,y)=((\frac{1}{a}(x_1+y_1-1+(a-1)x_1y_1)\vee0),((x_2+y_2-1)(1+\lambda)-\lambda x_2y_2)\vee0,1\wedge(x_3^b+y_3^b)^{\frac{1}{b}}),a\in(0,1],\lambda\geq0,b\geq1$,
* $T_8(x,y)=(((x_1+y_1-1)(1+\lambda)-\lambda x_1y_1)\vee0,(\frac{1}{a}(x_2+y_2-1+(a-1)x_2y_2)\vee0),1\wedge(x_3^b+y_3^b)^{\frac{1}{b}}),a\in(0,1],b\geq1,\lambda\geq0$,
* $T_9(x,y)=((\frac{1}{a}(x_1+y_1-1+(a-1)x_1y_1)\vee0),0\vee(x_2^b+y_2^b-1)^{\frac{1}{b}},1\wedge(x_3^c+y_3^c)^{\frac{1}{c}}),a\in(0,1],b,c\geq1$,
* $T_{10}(x,y)=(0\vee(x_1^a+y_1^a-1)^{\frac{1}{a}},(\frac{1}{b}(x_2+y_2-1+(b-1)x_2y_2)\vee0),1\wedge(x_3^c+y_3^c)^{\frac{1}{c}}),b\in(0,1],a,c\geq1$.
* $T_{11}(x,y)=(((x_1+y_1-1)(1+\lambda)-\lambda x_1y_1)\vee0,0\vee(x_2^a+y_2^a-1)^{\frac{1}{a}},1\wedge(x_3^b+y_3^b)^{\frac{1}{b}}),\lambda\geq0,a,b\geq1$,
* $T_{12}(x,y)=(0\vee(x_1^a+y_1^a-1)^{\frac{1}{a}},((x_2+y_2-1)(1+\lambda)-\lambda x_2y_2)\vee0,1\wedge(x_3^b+y_3^b)^{\frac{1}{b}}),\lambda\geq0,a,b\geq1$.

3. Nilpotent-nilpotent-strict subclass of t-norms, denoted by Δ_{nns}:

Definition 4.3. A picture fuzzy t-norm T is called nilpotent-nilpotent-strict iff:

$$T\left(x,y\right)=\left(t_1\left(x_1,y_1\right),t_2\left(x_2,y_2\right),s_3\left(x_3,y_3\right)\right),\forall x,y\in D^*,$$

where t_1,t_2 are nilpotent t-norms and s_3 is a strict t-conorm on [0,1].

Examples 4.3.

* $T_{13}(x,y)=(0\vee(x_1+y_1-1),0\vee(x_2+y_2-1),x_3+y_3-x_3y_3)$.
* $T_{14}(x,y)=(\frac{1}{2}(x_1+y_1-1+x_1y_1)\vee0,\frac{1}{2}(x_2+y_2-1+x_2y_2)\vee0,(x_3^a+y_3^a-x_3^ay_3^a)^{\frac{1}{a}}),a\geq1$,
* $T_{15}(x,y)=(((x_1+y_1-1)(1+\lambda_1)-\lambda_1x_1y_1)\vee0,((x_2+y_2-1)(1+\lambda_2)-\lambda_2x_2y_2)\vee0,(x_3^a+y_3^a-x_3^ay_3^a)^{\frac{1}{a}}),\lambda_1,\lambda_2\in[0,+\infty),a\geq1$,
* $T_{16}(x,y)=(0\vee(x_1^a+y_1^a-1)^{\frac{1}{a}},(x_2^b+y_2^b-1)^{\frac{1}{b}}\vee0,(x_3^c+y_3^c-x_3^cy_3^c)^{\frac{1}{c}}),a,b,c\geq1$,
* $T_{17}(x,y)=(\frac{1}{a}(x_1+y_1-1+(a-1)x_1y_1)\vee0,0\vee(x_2^b+y_2^b-1)^{\frac{1}{b}},(x_3^c+y_3^c-x_3^cy_3^c)^{\frac{1}{c}}),a\in(0,1];b,c\geq1$,
* $T_{18}(x,y)=(\frac{1}{a}(x_1+y_1-1+(a-1)x_1y_1)\vee0,\frac{1}{b}(x_2+y_2-1+(b-1)x_2y_2)\vee0,(x_3^c+y_3^c-x_3^cy_3^c)^{\frac{1}{c}}),a,b\in(0,1];c\geq1$,
* $T_{19}(x,y)=(\frac{1}{a}(x_1+y_1-1+(a-1)x_1y_1)\vee0,((x_2+y_2-1)(1+b)-bx_2y_2)\vee0,(x_3^c+y_3^c-x_3^cy_3^c)^{\frac{1}{c}}),a\in(0,1];b\geq0;c\geq1$,
* $T_{20}(x,y)=(0\vee(x_1^a+y_1^a-1)^{\frac{1}{a}},\frac{1}{b}(x_2+y_2-1+(b-1)x_2y_2)\vee0,(x_3^c+y_3^c-x_3^cy_3^c)^{\frac{1}{c}}),b\in(0,1];a,c\geq1$,
* $T_{21}(x,y)=(((x_1+y_1-1)(1+a)-ax_1y_1)\vee0,\frac{1}{b}(x_2+y_2-1+(b-1)x_2y_2)\vee0,(x_3^c+y_3^c-x_3^cy_3^c)^{\frac{1}{c}}),a\geq0;b\in(0,1];c\geq1$,
* $T_{22}(x,y)=(((x_1+y_1-1)(1+\lambda)-\lambda x_1y_1)\vee0,0\vee(x_2^a+y_2^a-1)^{\frac{1}{a}},(x_3^b+y_3^b-x_3^by_3^b)^{\frac{1}{b}}),\lambda\geq0,a,b\geq1$,
* $T_{23}(x,y)=(0\vee(x_1^a+y_1^a-1)^{\frac{1}{a}},((x_2+y_2-1)(1+\lambda)-\lambda x_2y_2)\vee0,(x_3^b+y_3^b-x_3^by_3^b)^{\frac{1}{b}}),\lambda\geq0,a,b\geq1$.

4. Strict-nilpotent-strict subclass of t-norms, denoted by Δ_{sns}:

Definition 4.4. A picture fuzzy t-norm T is called strict-nilpotent-strict iff:

$$T\left(x,y\right)=\left(t_1\left(x_1,y_1\right),t_2\left(x_2,y_2\right),s_3\left(x_3,y_3\right)\right),\forall x,y\in D^*,$$

where t_1 is a strict t-norm, t_2 is a nilpotent t-norm and s_3 is a strict t-conorm on [0,1].

Examples 4.4.

* $T_{24}(x,y)=(x_1y_1,0\vee(x_2+y_2-1),x_3+y_3-x_3y_3)$.

* $T_{25}(x,y) = (\frac{x_1y_1}{\lambda_1+(1-\lambda_1)(x_1+y_1-x_1y_1)}, ((x_2+y_2-1)(1+\lambda_2) - \lambda_2x_2y_2) \vee 0, (x_3^a+y_3^a-x_3^ay_3^a)^{\frac{1}{a}}), \lambda_1 \geq 1, \lambda_2 \geq 0, a \geq 1,$
* $T_{26}(x,y) = (\frac{x_1y_1}{\lambda_1+(1-\lambda_1)(x_1+y_1-x_1y_1)}, 0 \vee (x_2^a+y_2^a-1)^{\frac{1}{a}}, (x_3^b+y_3^b-x_3^by_3^b)^{\frac{1}{b}}), \lambda_1 \geq 1, a, b \geq 1,$
* $T_{27}(x,y) = (\frac{x_1y_1}{\lambda_1+(1-\lambda_1)(x_1+y_1-x_1y_1)}, \frac{1}{a}(x_2+y_2-1+(a-1)x_2y_2) \vee 0, (x_3^b+y_3^b-x_3^by_3^b)^{\frac{1}{b}}), \lambda_1, b \geq 1, a \in (0,1].$

5. Nilpotent-strict-strict subclass of t-norms, denoted by Δ_{nss}:

Definition 4.5. A picture fuzzy t-norm T is called nilpotent-strict-strict iff:

$$T(x,y) = (t_1(x_1,y_1), t_2(x_2,y_2), s_3(x_3,y_3)), \forall x,y \in D^*,$$

where t_1 is a nilpotent t-norm, t_2 is a strict t-norm and s_3 is a strict t-conorm on [0,1].

Examples 4.5.

* $T_{28}(x,y) = (0 \vee (x_1+y_1-1), x_2y_2, x_3+y_3-x_3y_3),$
* $T_{29}(x,y) = (\frac{1}{a}(x_1+y_1-1+(a-1)x_1y_1) \vee 0, \frac{x_2y_2}{\lambda+(1-\lambda)(x_2+y_2-x_2y_2)}, (x_3^b+y_3^b-x_3^by_3^b)^{\frac{1}{b}}), a \in (0,1]; b, \lambda \geq 1,$
* $T_{30}(x,y) = (0 \vee (x_1^a+y_1^a-1)^{\frac{1}{a}}, \frac{x_2y_2}{\lambda+(1-\lambda)(x_2+y_2-x_2y_2)}, (x_3^b+y_3^b-x_3^by_3^b)^{\frac{1}{b}}), a, b, \lambda \geq 1,$
* $T_{31}(x,y) = (((x_1+y_1-1)(1+\lambda_1) - \lambda_1x_1y_1, \frac{x_2y_2}{\lambda_2+(1-\lambda_2)(x_2+y_2-x_2y_2)}, (x_3^a+y_3^a-x_3^ay_3^a)^{\frac{1}{a}}), a, \lambda_2 \geq 1, \lambda_1 \in (0,1].$

Proposition 4.1. There does not exist any representable picture fuzzy t-norm of the form:

$$T(x,y) = (t_1(x_1,y_1), t_2(x_2,y_2), s_3(x_3,y_3)), \forall x,y \in D^*,$$

where t_1 or t_2 is a strict t-norm, and s_3 is a nilpotent t-conorm on [0,1].

Proposition 4.2. If T belongs to one of four subclasses Δ_{sss}, Δ_{nns}, Δ_{sns}, Δ_{nss} then T is a strict t-norm.

Proposition 4.3. If T belongs to the subclass Δ_{nnn} then T is a nilpotent t-norm.

V. A CLASSIFICATION OF REPRESENTABLE PICTURE T-CONORMS

We have some subclasses of representable picture fuzzy t-conorms as follows:

1. Strict-strict-strict subclass of t-conorms, denoted by ∇_{sss}:

Definition 5.1. A picture fuzzy t-conorm S is called strict-strict-strict iff:

$$S(x,y) = (s_3(x_1,y_1), t_2(x_2,y_2), t_1(x_3,y_3)), \forall x,y \in D^*,$$

where t_1, t_2 are strict fuzzy t-norms and s_3 is a strict fuzzy t-conorm on [0,1].

Example 5.1.

* $S_1(x,y) = (x_1+y_1-x_1y_1, x_2y_2, x_3y_3),$
* $S_2(x,y) = ((x_1^a+y_1^a-x_1^ay_1^a)^{\frac{1}{a}}, \frac{x_2y_2}{\lambda_1+(1-\lambda_1)(x_2+y_2-x_2y_2)}, \frac{x_3y_3}{\lambda_2+(1-\lambda_2)(x_3+y_3-x_3y_3)}), \lambda_1, \lambda_2, a \in [1,+\infty),$

2. Nilpotent-nilpotent-nilpotent subclass of t-conorms, denoted by ∇_{nnn}:

Definition 5.2. A picture fuzzy t-conorm S is called nilpotent-nilpotent-nilpotent iff:

$$S(x,y) = (s_3(x_1,y_1), t_2(x_2,y_2), t_1(x_3,y_3)), \forall x,y \in D^*,$$

where t_1, t_2 are nilpotent fuzzy t-norms and s_3 is a nilpotent fuzzy t-conorm on [0,1].

Example 5.2.

* $S_3(x,y) = (1 \wedge (x_1+y_1), 0 \vee (x_2+y_2-1), 0 \vee (x_3+y_3-1)),$
* $S_4(x,y) = (1 \wedge (x_1^a+y_1^a)^{\frac{1}{a}}, ((x_2+y_2-1)(1+\lambda_1) - \lambda_1x_2y_2) \vee 0, ((x_3+y_3-1)(1+\lambda_2) - \lambda_2x_3y_3) \vee 0), \lambda_1, \lambda_2 \in [0,+\infty), a \geq 1,$
* $S_5(x,y) = (1 \wedge (x_1^a+y_1^a)^{\frac{1}{a}}, (0 \vee (x_2^b+y_2^b-1))^{\frac{1}{b}}, 0 \vee (x_3^c+y_3^c-1)^{\frac{1}{c}}), a, b, c \geq 1,$
* $S_6(x,y) = (1 \wedge (x_1^a+y_1^a)^{\frac{1}{a}}, (\frac{1}{b}(x_2+y_2-1+(b-1)x_2y_2) \vee 0), (\frac{1}{c}(x_3+y_3-1+(c-1)x_3y_3) \vee 0)), a \geq 1; b, c \in (0,1],$
* $S_7(x,y) = (1 \wedge (x_1^a+y_1^a)^{\frac{1}{a}}, (\frac{1}{b}(x_2+y_2-1+(b-1)x_2y_2) \vee 0), ((x_3+y_3-1)(1+\lambda) - \lambda x_3y_3) \vee 0), a \geq 1, b \in (0,1], \lambda \geq 0,$
* $S_8(x,y) = (1 \wedge (x_1^a+y_1^a)^{\frac{1}{a}}, ((x_2+y_2-1)(1+\lambda) - \lambda x_2y_2) \vee 0, (\frac{1}{b}(x_3+y_3-1+(b-1)x_3y_3) \vee 0)), a \geq 1, b \in (0,1], \lambda \geq 0,$
* $S_9(x,y) = (1 \wedge (x_1^a+y_1^a)^{\frac{1}{a}}, (\frac{1}{b}(x_2+y_2-1+(b-1)x_2y_2) \vee 0), 0 \vee (x_3^c+y_3^c-1)^{\frac{1}{c}}), b \in (0,1], a, c \geq 1,$
* $S_{10}(x,y) = (1 \wedge (x_1^a+y_1^a)^{\frac{1}{a}}, 0 \vee (x_2^b+y_2^b-1)^{\frac{1}{b}}, (\frac{1}{c}(x_3+y_3-1+(c-1)x_3y_3) \vee 0)), c \in (0,1], a, b \geq 1.$
* $S_{11}(x,y) = (1 \wedge (x_1^a+y_1^a)^{\frac{1}{a}}, ((x_2+y_2-1)(1+\lambda) - \lambda x_2y_2) \vee 0, 0 \vee (x_3^b+y_3^b-1)^{\frac{1}{b}}), \lambda \geq 0, a, b \geq 1,$
* $S_{12}(x,y) = (1 \wedge (x_1^a+y_1^a)^{\frac{1}{a}}, 0 \vee (x_2^b+y_2^b-1)^{\frac{1}{b}}, ((x_3+y_3-1)(1+\lambda) - \lambda x_3y_3) \vee 0), \lambda \geq 0, a, b \geq 1.$

3. Strict-nilpotent-nilpotent subclass of t-conorms, denoted by ∇_{snn}:

Definition 5.3. A picture fuzzy t-conorm S is called strict-nilpotent-nilpotent iff:

$$S(x,y) = (s_3(x_1,y_1), t_2(x_2,y_2), t_1(x_3,y_3)), \forall x,y \in D^*,$$

where t_1, t_2 are nilpotent fuzzy t-norms and s_3 is a strict fuzzy t-conorm on [0,1].

Example 5.3.

* $S_{13}(x,y) = (x_1+y_1-x_1y_1, 0 \vee (x_2+y_2-1), 0 \vee (x_3+y_3-1)),$
* $S_{14}(x,y) = ((x_1^a+y_1^a-x_1^ay_1^a)^{\frac{1}{a}}, \frac{1}{2}(x_2+y_2-1+x_2y_2) \vee 0, \frac{1}{2}(x_3+y_3-1+x_3y_3) \vee 0), a \geq 1,$
* $S_{15}(x,y) = ((x_1^a+y_1^a-x_1^ay_1^a)^{\frac{1}{a}}, ((x_2+y_2-1)(1+\lambda_1) - \lambda_1x_2y_2) \vee 0, ((x_3+y_3-1)(1+\lambda_2) - \lambda_2x_3y_3) \vee 0), \lambda_1, \lambda_2 \in [0,+\infty), a \geq 1,$
* $S_{16}(x,y) = ((x_1^c+y_1^c-x_1^cy_1^c)^{\frac{1}{c}}, 0 \vee (x_2^a+y_2^a-1)^{\frac{1}{a}}, 0 \vee (x_3^b+y_3^b-1)^{\frac{1}{b}}), a, b, c \geq 1,$
* $S_{17}(x,y) = ((x_1^a+y_1^a-x_1^ay_1^a)^{\frac{1}{a}}, \frac{1}{b}(x_2+y_2-1+(b-1)x_2y_2) \vee 0, 0 \vee (x_3^c+y_3^c-1)^{\frac{1}{c}}), b \in (0,1]; a, c \geq 1,$

* $S_{18}(x,y) = ((x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, \frac{1}{b}(x_2 + y_2 - 1 + (b-1)x_2y_2) \vee 0, \frac{1}{c}(x_3 + y_3 - 1 + (c-1)x_3y_3) \vee 0), b, c \in (0,1]; a \geq 1,$
* $S_{19}(x,y) = ((x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, \frac{1}{b}(x_2 + y_2 - 1 + (b-1)x_2y_2) \vee 0, ((x_3 + y_3 - 1)(1+c) - cx_3y_3) \vee 0), a \geq 1, b \in (0,1]; c \geq 0,$
* $S_{20}(x,y) = ((x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, (x_2^b + y_2^b - 1)^{\frac{1}{b}} \vee 0, \frac{1}{c}(x_3 + y_3 - 1 + (c-1)x_3y_3) \vee 0), c \in (0,1]; a, b \geq 1,$
* $S_{21}(x,y) = ((x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, ((x_2 + y_2 - 1)(1+b) - bx_2y_2) \vee 0, \frac{1}{c}(x_3 + y_3 - 1 + (c-1)x_3y_3) \vee 0), a \geq 1; b \geq 0; c \in (0,1],$
* $S_{22}(x,y) = ((x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, ((x_2 + y_2 - 1)(1+\lambda) - \lambda x_2y_2) \vee 0, 0 \vee (x_3^b + y_3^b - 1)^{\frac{1}{b}}), \lambda \geq 0, a, b \geq 1,$
* $S_{23}(x,y) = ((x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, 0 \vee (x_2^b + y_2^b - 1)^{\frac{1}{b}}, ((x_3 + y_3 - 1)(1+\lambda) - \lambda x_3y_3) \vee 0), \lambda \geq 0, a, b \geq 1.$

4. Strict-nilpotent-strict subclass of t-conorms, denoted by ∇_{sns}:

Definition 5.4. A picture fuzzy t-conorm S is called strict-nilpotent-strict iff:

$$S(x,y) = (s_3(x_1,y_1), t_2(x_2,y_2), t_1(x_3,y_3)), \forall x, y \in D^*,$$

where t_1 is a strict fuzzy t-norm, t_2 is a nilpotent fuzzy t-norm and s_3 is a strict fuzzy t-conorm on [0,1].

Example 5.4.

* $S_{24}(x,y) = (x_1 + y_1 - x_1y_1, 0 \vee (x_2 + y_2 - 1), x_3y_3),$
* $S_{25}(x,y) = ((x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, ((x_2 + y_2 - 1)(1+\lambda_1) - \lambda_1 x_2y_2) \vee 0, \frac{x_3y_3}{\lambda_2 + (1-\lambda_2)(x_3+y_3-x_3y_3)}), \lambda_1 \geq 0; \lambda_2, a \geq 1,$
* $S_{26}(x,y) = ((x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, 0 \vee (x_2^b + y_2^b - 1)^{\frac{1}{b}}, \frac{x_3y_3}{\lambda_1 + (1-\lambda_1)(x_3+y_3-x_3y_3)}), \lambda_1 \geq 1; a, b \geq 1,$
* $S_{27}(x,y) = ((x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, \frac{1}{b}(x_2 + y_2 - 1 + (b-1)x_2y_2) \vee 0, \frac{x_3y_3}{\lambda_1 + (1-\lambda_1)(x_3+y_3-x_3y_3)}), a, \lambda_1 \geq 1; b \in (0,1].$

5. Strict-strict-nilpotent subclass of t-conorms, denoted by ∇_{ssn}:

Definition 5.5. A picture fuzzy t-conorm S is called strict-strict-nilpotent iff:

$$S(x,y) = (s_3(x_1,y_1), t_2(x_2,y_2), t_1(x_3,y_3)), \forall x, y \in D^*,$$

where t_1 is a nilpotent t-norm, t_2 are strict fuzzy t-norms, and s_3 is a strict fuzzy t-conorm on [0,1].

Example 5.5.

* $S_{28}(x,y) = (x_1 + y_1 - x_1y_1, x_2y_2, 0 \vee (x_3 + y_3 - 1)),$
* $S_{29}(x,y) = ((x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, \frac{x_2y_2}{\lambda + (1-\lambda)(x_2+y_2-x_2y_2)}, \frac{1}{b}(x_3 + y_3 - 1 + (b-1)x_3y_3) \vee 0), a, \lambda \geq 1; b \in (0,1],$
* $S_{30}(x,y) = ((x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, \frac{x_2y_2}{\lambda + (1-\lambda)(x_2+y_2-x_2y_2)}, 0 \vee (x_3^b + y_3^b - 1)^{\frac{1}{b}}), a, b, \lambda \geq 1,$
* $S_{31}(x,y) = ((x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, \frac{x_2y_2}{\lambda + (1-\lambda)(x_2+y_2-x_2y_2)}, ((x_2 + y_2 - 1)(1+b) - bx_2y_2) \vee 0), a, \lambda \geq 1; b \geq 0.$

Proposition 5.1. There does not exists representable picture fuzzy t-conorm S iff:

$$S(x,y) = (s_3(x_1,y_1), t_2(x_2,y_2), t_1(x_3,y_3)), \forall x, y \in D^*,$$

where t_1 or t_2 is a strict t -norm and s_3 is a nilpotent fuzzy t-conorm on [0,1].

Proposition 5.2. If S belongs to one of four subclasses ∇_{sss}, ∇_{snn}, ∇_{sns}, ∇_{ssn} then S is a strict t-conorm.

Proposition 5.3. If S belongs to the subclass ∇_{nnn} then S is a nilpotent t-conorm.

VI. Conclusion

In this paper picture fuzzy theory have been continuously constructed with proposing the concepts which be extended from fuzzy cases: picture fuzzy negators N and the representation theory on them, picture fuzzy t-norms T, picture fuzzy t-conorms S. In the our paper [7, 8] some new properties of an involutive fuzzy negator and some corresponding De Morgan picture fuzzy triples have been presented. In this paper some classifications of reprentable picture t-norms operators and picture t-conorms operators are defined. Future new concepts should be devoted to the picture fuzzy soft sets. The compositional rule of inference in picture fuzzy logic setting should be presented in the next papers.

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