Z-Numbers and Type-2 Fuzzy Sets: A Representation Result

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Abstract

Traditional $[0,1]$-based fuzzy sets were originally invented to describe expert knowledge expressed in terms of imprecise ("fuzzy") words from natural language. To make this description more adequate, several generalizations of the traditional $[0,1]$-based fuzzy sets have been proposed, among them type-2 fuzzy sets and Z-numbers. The main objective of this paper is to study the relation between these two generalizations. As a result of this study, we show that if we apply data processing to Z-numbers, then we get type-2 sets of special type – that we call *monotonic*. We also prove that every monotonic type-2 fuzzy set can be represented as a result of applying an appropriate data processing algorithm to some Z-numbers.

1 Z-Numbers and Type-2 Fuzzy Sets: Formulation of the Problem

Need for motivations. In this paper, we formulate – and answer – a question about the relation between Z-numbers and type-2 fuzzy sets, two generalizations of the traditional fuzzy sets. To understand why this question is important, let us first recall why we need fuzzy sets – both traditional and generalized – in the first place: this need comes from the need to formalize expert knowledge.

Need to describe expert knowledge in computer-understandable terms. In many application areas, we reply on human expertise: when we want to fly to a conference, we rely on a pilot; when we get sick, we go to a doctor, etc.

Some experts are better than others. In the ideal world, we should all be served by the best experts: every plane should be controlled by the most skilled pilot, every patient should be treated by the best medical doctor. In practice, however, a few best doctors cannot cure all the patients, and a few most skilled pilots cannot navigate all the planes.

It is therefore important to design computer-based systems that will incorporate the knowledge and skills of the best experts and thus, help other experts make better decisions. For that, we need to describe the expert knowledge in computer-understandable terms

Need for fuzzy logic. Some of the experts' knowledge is precise and thus, easy to describe in computer-understandable terms. For example, one can easily describe, in such terms, a medical doctor's recommendation that any patient with a body temperature of 38°C or higher will be given a dose of aspirin proportional to his/her body weight.

However, many expert rules are not that precise. For example, instead of specifying a 38°C threshold, a medical doctor may say that a patient with high fever be given aspirin – without explicitly specifying what "high fever" means.

Rules and statements using such imprecise ("fuzzy") words from natural language like "high" are ubiquitous in our knowledge. To describe such knowledge in precise terms, Lotfi Zadeh invented a special technique that he called *fuzzy logic*; see, e.g., [2, 7, 8]. According to this technique, to describe the meaning of each imprecise term like "high", we ask the expert to describe, for each possible value x of the corresponding quantity (e.g., temperature) the degree $\mu(x) \in [0,1]$ to which this value can be characterized by this term, so that 0 means absolutely not high, 1 means absolutely high, and intermediate values mean somewhat high. One way to get each value $\mu(x)$ is ask an expert to indicate his/her degree by a point on a scale – e.g., on a scale from 0 to 10. If an expert marks 37.9°C as corresponding to 7 on this scale, then we describe his/her degree of 37.9°C being "high" by the ratio 7/10.

A function assigning, to each possible value x, the corresponding degree $\mu(x)$, is known as a *membership function* or, alternatively, as a *fuzzy set*.

Comment. Note that, in general, the quantity x does not need to be number-valued: alternatively, its values can be, e.g., vectors.

Need for "and"- and "or"-operations. Many expert rules involve several conditions. For example, since some efficient fever-lowering medicines increase blood pressure, a medical doctor may recommend the corresponding medicine is the fever is high *and* the blood pressure is not high.

Ideally, we should consider all possible pairs (x, y) of temperature and blood pressure, and for each such pair, elicit, from the expert, his/her degree that the corresponding "and"-condition is satisfied. However, in practice, there is a large number of such combinations, so it may not be possible to ask the expert's opinion about all of them. This is especially true if we take into account that sometimes, expert rules include three, four (and even more) conditions – in this case, asking the expert about all such combinations is plainly impossible.

In such situations, since we cannot elicit the expert's degree of confidence about a composite statement $A \,\&\, B$, we have to estimate this degree based on the known degree of confidence a and b in the components statements A and B. In our example, A is the statement that x is a high temperature, and B is the statement that y is not a high blood pressure.

The corresponding estimate depends only on a and b, so it has the form $f_\&(a,b)$ for an appropriate algorithmic function $f_\&$. This function is known as an *"and"-operation* or a *t-norm.* Similarly, to estimate the expert's degree of confidence in a statement $A \vee B$, we need an *"or"-operation* $f_\vee(a,b)$; "or"-operations are also known as *t-conorms*.

The corresponding operations should satisfy some reasonable properties. For example, since $A \,\&\, B$ is equivalent to $B \,\&\, A$, it makes sense to require that the "and"-operation provide the same estimate for both expressions – i.e., that it be commutative: $f_\&(a,b) = f_\&(b,a)$. Similarly, since $A \,\&\, (B \,\&\, C)$ is equivalent to $(A \,\&\, B) \,\&\, C$, the "and"-operation must be associative, etc.

The simplest operations that satisfy all these properties are $f_\&(a,b) = \min(a,b)$ and $f_\vee(a,b) = \max(a,b)$. These operations are among the most widely used in applications of fuzzy techniques [2, 7, 8].

Processing fuzzy data: Zadeh's extension principle. In case of precise rules, we use the values of the inputs $x_1, \ldots, x_n$ to determine the values of the desired quantity y – e.g., the value of the parameter that describes the appropriate control. Let us denote the corresponding algorithmic function by $y = f(x_1, \ldots, x_n)$. Computing the corresponding values y is an important part of *data processing.*

If instead of measured values, we use expert estimates, then instead of the values x_i of the corresponding quantities X_i, we have fuzzy sets $\mu_i(x_i)$ that describe our knowledge about these quantities. In such situations, it is desirable to come up with a similar description for the possible values y of the desired quantity Y.

A number y is a possible value of the quantity Y if there exists a tuple of values $(x_1, \ldots, x_n)$ for which $y = f(x_1, \ldots, x_n)$ and each x_i is a possible value of the corresponding quantity X_i. For each number x_i, the degree to which this number is a possible value of the quantity X_i is equal to $\mu_i(x_i)$. Thus, if we use the min "and"-operation, the degree to which x_1 is a possible value of X_1 *and* x_2 is a possible value of X_2, etc., is equal to $\min(\mu_1(x_1), \ldots, \mu_n(x_n))$.

The condition $y = f(x_1, \ldots, x_n)$ is either absolutely true (i.e., has degree 1) or absolutely false (degree 0). Thus, the degree to which each x_i is a possible value of X_i and $y = f(x_1, \ldots, x_n)$ is equal to $\min(\mu_1(x_1), \ldots, \mu_n(x_n))$ when $y = f(x_1, \ldots, x_n)$ and to 0 otherwise.

The phrase "there exists a tuple" means that either the corresponding property holds for one tuple, or for another tuple, etc. If we use the simplest max "or"-operation, then the degree $\mu(y)$ to which y is a possible value of Y takes the following form:

$$\mu(y) = \max\{\min(\mu_1(x_1), \ldots, \mu_n(x_n)) : y = f(x_1, \ldots, x_n)\}.$$

This formula – first proposed by Zadeh – extends functions from real numbers to fuzzy inputs and is thus known as *Zadeh's extension principle* [2, 7, 8].

Comment. In describing the degree of confidence $\mu(y)$, in principle, we can use a different "and"-operation, e.g., the algebraic product $f_\&(a,b) = a \cdot b$.

However, we do not have much choice with the "or"-operation. Indeed, if instead of $f_\vee(a,b) = \max(a,b)$, we use, e.g., the algebraic sum $f_\vee(a,b) = a + b - a \cdot b$, then the "or"-combination of infinitely many degrees will lead to a meaningless $\mu(y) = 1$ for all y.

Type-2 fuzzy sets and Z-numbers. The traditional $[0,1]$-based fuzzy techniques are based on the implicit assumption that an expert can always describe his/her degree of confidence in a statement by a number. In practice, this may be difficult: an expert may be able to meaningfully distinguish between 7 and 8 on a 0-to-10 scale, but hardly anyone can differentiate between, say 7.0 and 7.1 on this scale. In other words, instead of selecting a single number, an expert may be more comfortable selecting several numbers – maybe with the degree to which each of these numbers describes his/er opinion.

So, for each value x, the expert describes, for each possible degree μ, a degree $d(x,\mu)$ to which μ is a reasonable degree of x being high. Thus, the degree $\mu(x)$ characterizing the expert's opinion about the value x is no longer a number, it is itself a fuzzy set. Membership functions that assign such a fuzzy set to each value x are known as *type-2* fuzzy set; see, e.g., [4, 5, 6].

Another generalization of the traditional fuzzy sets is related to the fact that experts are often not 100% confident in their degrees. So, in addition to eliciting a degree $\mu(x)$, it makes sense to also elicit the degree $\nu(x)$ to which the expert is certain in his/her evaluation. The corresponding pairs $(\mu(x), \nu(x))$ is known as a *Z-number*, after L. Zadeh [1, 9].

Comment. Note that this is one possible definition of a Z-number; different formalization of the original Zadeh's idea of a Z-number may lead to slightly different definitions. For example, in [3], we used a single degree $\nu(x) = \text{const}$ to describe the expert's degree of confidence for all x. In this paper, we consider a more general definition, in which we allow the degrees $\nu(x)$ to depend on x.

Main question: what is the relation between Z-numbers and type-2 fuzzy sets? For both generalizations, instead of a single value $\mu(x)$, we have degrees describing to what extend different degrees are possible. In other words, while the meanings of two extensions are different, from the purely mathematical viewpoint, these two extensions seem similar. So what is the relation between the two extensions?

What we do in this paper. In this paper, we explain the relation between Z-numbers and type-2 fuzzy sets. Specifically, we prove that if we apply data processing to Z-numbers, then we get type-2 fuzzy sets of a special type – which we will call *monotonic*, and that, vice versa, every monotonic type-2 fuzzy set can be represented as a result of applying some data processing algorithm to appropriate Z-numbers.

2 The Result of Applying Data Processing to Z-Numbers Is a Monotonic Type-2 Fuzzy Set

Need to consider the result of applying data processing to Z-numbers. In the usual Zadeh's extension principle, when we apply the data processing algorithm $y = f(x_1, \ldots, x_n)$, we assume that for each i and for each possible value x_i of the quantity X_i, we know the degree $\mu_i(x_i)$ to which this value x_i is possible.

In the Z-number case, in addition to each degree $\mu_i(x_i)$, we also know the degree $\nu_i(x_i)$ to which the expert is confident in the degree $\mu_i(x_i)$. How will this additional information affect the result of data processing?

Need to consider "and"- and "or"-operations for Z-numbers. In our derivation of Zadeh's extension principle, we used the "and" and "or"-operations – namely, min and max. To extend Zadeh's extension principle to Z-numbers, it is therefore necessary to extend the usual "and"- and "or"-operations to Z-numbers.

How to extend "and"-operations to Z-numbers. Let us assume that the expert's degree of confidence in a statement A is a and the expert's degree of confidence in this estimate is ν_a. Let us also assume that the expert's degree of confidence in a statement B is b and the expert's degree of confidence in this estimate is ν_b.

Then, the expert's degree of confidence in a composite statement $A \,\&\, B$ is $f_\&(a, b)$. If we use the simplest possible "and"-operation $f_\&(a, b) = \min(a, b)$, then this degree is equal to $\min(a, b)$.

What is the expert's degree of confidence in the estimate $\min(a, b)$? This estimate makes sense only if both estimates a and b make sense. In other words, an expert is confident in the combined estimate $\min(a, b)$ if the expert is confident in the estimate a *and* confident in estimate b. So, the expert's degree of confidence in the combined estimate $\min(a, b)$ can be obtained by applying the "and"-operation to the degrees of confidence ν_a and μ_b in both estimates: $f_\&(\mu_a, \mu_b)$. In particular, if we use the min "and"-operation, we get the degree $\min(\nu_a, \nu_b)$.

Comment. The degrees ν_a and ν_b are usually viewed as subjective probabilities, with the "and"-operation $f_\&(a, b) = a \cdot b$. In general, this is OK, but, as we will see, the choice of the algebraic product "and"-operation lead to meaningless 0 values for the results of data processing – similar to the fact that the use of the algebraic sum leads to meaningless 1 for the usual Zadeh's extension principle. To avoid such meaningless values, in this paper, we use the min "and"-operation.

How to extend "or"-operations to Z-numbers. What about the "or"-operation? What is the expert's degree of confidence in the corresponding estimate $f_\vee(a, b) = \max(a, b)$? At first glance, by analogy, it may seem that we get $\max(\nu_a, \nu_b)$, but a more detailed analysis shows that this is a wrong formula. Indeed, an expert is confident in the combined estimate $\max(a, b)$ if the expert is confident in the estimate a *and* confident in estimate b. So, similar to the case of the "and"-operations, the expert's degree of confidence in the combined estimate $\min(a, b)$ can be obtained by applying the "and"-operation to the degrees of confidence ν_a and μ_b in both estimates: $f_\&(\mu_a, \mu_b)$. In particular, if we use the min "and"-operation, we get the degree $\min(\nu_a, \nu_b)$ – the same degree as for the "and"-operation.

Now, we are ready to describe a natural way to generalize Zadeh's extension principle to Z-numbers.

How to generalize Zadeh's extension principle to Z-numbers: formulation of the problem. Let us now consider the case when all the inputs to a data processing algorithm $y = f(x_1, \ldots, x_n)$ are Z-numbers, i.e., that for each input i and for each possible value x_i of the i-th quantity X_i, we know not only the expert's degree of confidence $\mu_i(x_i)$ that x_i is a possible value of X_i, but also the expert's degree of confidence $\nu_i(x_i)$ in this estimate.

Based on this information, what can we say about the possible values y of the desired quantity Y?

How to generalize Zadeh's extension principle to Z-numbers: seemingly natural approach and its limitations. In line with the above description of "and"- and "or"-operations for Z-numbers, for each tuple $(x_1, \ldots, x_n)$ for which $y = f(x_1, \ldots, x_n)$, the resulting value y is possible with degree $\min(\mu_1(x_1), \ldots, \mu_n(x_n))$, and the expert's confidence in this estimate is equal to $\min(\nu_1(x_1), \ldots, \nu_n(x_n))$.

In principle, we could do what we did when we derived Zadeh's extension principle, and for each y, simply combine the estimates corresponding to all the tuples $(x_1, \ldots, x_n)$ for which $y = f(x_1, \ldots, x_n)$. As a result, we would get the same degree $\mu(y)$ as in the traditional $[0, 1]$-based fuzzy case, but the problem is that the expert's confidence in this estimate would then be equal to

$$\nu(y) = \min\{\min(\nu_1(x_1), \ldots, \nu_n(x_n)) : y = f(x_1, \ldots, x_n)\}.$$

Since some of the degrees $\nu_i(x_i)$ may be very low, we will get the degree $\nu(y)$ very low – or even equal to 0. This means that the expert's confidence in the degree $\mu(y)$ is very low.

It make no sense to produce an estimate $\mu(y)$ in which the expert is not confident at all. Thus, we need to modify our approach.

Comment. The above formula shows that to compute the ν-degrees, we do not have much of a choice in selecting an "and"-operation. Indeed, if instead of min, we use, e.g., the algebraic product "and"-operation $f_\&(a, b) = a \cdot b$, then by taking the product of infinitely many degrees corresponding to infinitely many tuples, we will have a meaningless value $\mu(y) = 0$ always, even if – as we propose in the following text – we do not consider tuples with small value $\nu_i(x_i)$.

How to generalize Zadeh's extension principle to Z-numbers: analysis of the problem. We do not want to have an estimate with degree of confidence 0. Let us therefore select the desired degree of confidence $\nu > 0$, and let us try to come up with an estimate $\mu(y)$ for which the expert's degree of confidence $\mu(y)$ is at least as large as this threshold value: $\nu(y) \geq \nu$.

In other words, the minimum of the degrees of confidence corresponding to different tuples $(x_1, \ldots, x_n)$ must be at least μ. This is equivalent to saying that all these degrees of confidence must be at least μ. In other words, we should only consider tuples $(x_1, \ldots, x_n)$ for which $\min(\nu_1(x_1), \ldots, \nu_n(x_n)) \geq \nu$. This inequality, in its turn, is equivalent to requiring that $\nu_i(x_i) \geq \nu$ for each i. Thus, we arrive at the following definition.

How to generalize Zadeh's extension principle to Z-numbers: result. Let us assume that for each input i and for each possible value x_i of the i-th quantity X_i, we know the expert's degree of confidence $\mu_i(x_i)$ that x_i is a possible value of X_i, and the expert's degree of confidence $\nu_i(x_i)$ in this estimate.

Then, for each value $\nu \in [0, 1]$, we compute

$$\mu_\nu(y) = \max\{\min(\mu_1(x_1), \ldots, \mu_n(x_n)) : y = f(x_1, \ldots, x_n) \text{ and}$$

$$\nu_i(x_i) \geq \nu \text{ for all } i\}.$$

The function that assigns, to each $\nu \in [0, 1]$, the corresponding value $\mu_\nu(y)$, is the result of applying the data processing algorithm to Z-numbers – i.e., it is the desired extension of Zadeh's extension principle to Z-numbers.

Let us describe this in precise terms.

Definition 1. *By a* Z-number*, we mean a mapping that assigns, to every element x of a universal set, two numbers $mu(x)$ and $\nu(x)$ from the interval* $[0, 1]$*. We will say that:*

- *$\mu(x)$ is the expert's degree of confidence that x is a possible value, and*
- *$\nu(x)$ is the the expert's degree of confidence in the estimate $\mu(x)$.*

Definition 2. *Let $F : U_1 \times \ldots \times U_n \to U$ be a function, and for each $i = 1, \ldots, n$, let X_i be a Z-number defined on the universal set U_i. By the* result $f(X_1, \ldots, X_n)$ of applying the function $f(x_1, \ldots, x_n)$ to the Z-numbers $X_1, \ldots, X_n$*, we mean a function that assigns, to each $\nu \in [0, 1]$, the value*

$$\mu_\nu(y) = \max\{\min(\mu_1(x_1), \ldots, \mu_n(x_n)) : y = f(x_1, \ldots, x_n) \textit{ and}$$

$$\nu_i(x_i) \geq \nu \textit{ for all } i\}.$$

Comment. This is the desired extension of Zadeh's extension principle to Z-numbers.

In effect, the result of applying data processing to Z-numbers is a type-2 fuzzy set. Let us recall that we get a type-2 fuzzy set if instead of a single value $\mu(x)$, we get a function that assigns, to each value $\mu \in [0, 1]$, a degree $d(\mu, x) \in [0, 1]$.

Here, we have exactly this situation: to each value ν, we assign, a degree $\mu_\nu(x)$. Thus, from the purely mathematical viewpoint, *the result of applying data processing to Z-numbers is a type-2 fuzzy set.*

Can every type-2 fuzzy number be so represented? We have shown that the result of applying data processing to Z-numbers is a type-2 fuzzy set. A natural question is: can every type-2 fuzzy number be thus represented?

In the following text, we will prove that this is not the case: namely, that the type-2 fuzzy sets which are obtained as a result of applying data processing to Z-numbers have an additional property – that we will call *monotonicity.*

Monotonicity property. When the value ν increases, fewer and fewer tuples $(x_1, \ldots, x_n)$ satisfy the inequalities $\nu_i(x_i) \geq nu$. Thus, the maximum in the definition of $\mu_\nu(y)$ is over a smaller set of values – and, is thus, in general, smaller. In other words, if $\nu < \nu'$, then $\mu_\nu(x) \leq \mu_{\nu'}(x)$.

In this paper, we will call type-2 fuzzy sets with this property *monotonic.*

A natural question. We started with a question of whether every type-2 fuzzy number can be represented as a result of applying data processing to Z-numbers. We have shown that this is not the case, by proving that a type-2 fuzzy numbers obtained as a result of applying data processing to Z-numbers is always monotonic.

A natural next question is: Can every *monotonic* type-2 fuzzy set be represented as a result of applying data processing to Z-numbers? Our – positive – answer to this question is provided in the next section.

3 Every Monotonic Type-2 Fuzzy Set Can Be Represented as a Result of Applying Data Processing to Z-Numbers

Definition 3. *We say that a type-2 fuzzy number $d(y, \mu)$ is* monotonic *if $\mu < \mu'$ implies $d(y, \mu) \geq d(y, \mu')$.*

Representation Theorem. *Every monotonic type-2 fuzzy set $d(y, \mu)$ can be represented as a result of applying an appropriate data processing algorithm $y = f(x_1, \ldots, x_n)$ to some Z-numbers $X_1, \ldots, X_n$.*

Proof. Let us assume that we have a monotonic type-2 fuzzy set $d(y, \mu)$, where y takes all the values from some universal set, and $\mu \in [0, 1]$.

To construct the desired representation, we will take $n = 1$ and

$$U_1 = U \times [0, 1].$$

Each element $x_1 \in U_1$ is thus a pair $x_1 = (x_{11}, x_{12})$, where $x_{11} \in U$ and $x_{12} \in [0, 1]$.

Let us take the following data processing algorithm:

$$f(x_1) = f((x_{11}, x_{12})) = x_{11},$$

and let us take the following Z-number:

$$\mu_1(x_1) = \mu_1(x_{11}, x_{12})) = d(x_{11}, x_{12}),$$

and

$$\nu_1(x_1) = \nu((x_{11}, x_{12})) = x_{12}.$$

For this selection, since $n = 1$, the result $f(X_1)$ of applying the selected function $f(X_1)$ to the selected Z-number X_1 takes the form

$$\mu_\nu(y) = \max\{\mu_1(x_1) : f(x_1) = y \text{ and } \nu_1(x_1) \ge \nu\}.$$

By our choice of the data processing function $f(x_1)$, the condition $f(x_1) = y$ means that $x_{11} = y$. Thus $x_1 = (x_{11}, x_{12}) = (y, x_{12})$.

Similarly, the condition $\nu_1(x_1) \ge \nu$ means that $x_{12} \ge \nu$, and the value $\mu_1(x_1)$ is equal to $d(y, x_{12})$.

Hence, $\mu_\nu(y)$ is the maximum of all the values $d(y, x_{12})$ corresponding to all possible values $x_{12} \ge \nu$.

Since the type-2 fuzzy set is monotonic, the largest possible value $\mu_\nu(y)$ of $d(y, x_{12})$ is attained when x_{12} is the smallest possible, i.e., when $x_{12} = \nu$. Therefore, this largest value $\mu_\nu(y)$ is equal to $d(y, \nu)$. Thus, indeed, the given monotonic type-2 fuzzy set can be represented as the result $f(X_1)$ of applying the data processing algorithm $f(x_1)$ to the Z-number X_1. The theorem is proven.

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