For Multi-Interval-Valued Fuzzy Sets, Centroid Defuzzification Is Equivalent to Defuzzifying Its Interval Hull: A Theorem

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Abstract. In the traditional fuzzy logic, the expert's degree of certainty in a statement is described either by a number from the interval $[0, 1]$ or by a subinterval of such an interval. To adequately describe the opinion of several experts, researchers proposed to use a union of the corresponding sets – which is, in general, more complex than an interval. In this paper, we prove that for such set-valued fuzzy sets, centroid defuzzification is equivalent to defuzzifying its interval hull.
As a consequence of this result, we prove that the centroid defuzzification of a *general* type-2 fuzzy set can be reduced to the easier-to-compute case when for each x, the corresponding fuzzy degree of membership is *convex*.

1 Formulation of the Problem

Outline of this section. Our main objective is to come up with a centroid defuzzification formula for multi-interval-valued fuzzy sets. Before we start describing our results and algorithms, let us briefly recall why we need centroid defuzzification and why we need multi-interval-valued fuzzy sets. To explain this need:

- we will start with the regular fuzzy sets,
- then we explain the need for interval-valued fuzzy sets, and
- the need for multi-interval-valued fuzzy sets;
- finally, we explain the need for centroid defuzzification for all these types of fuzzy sets.

Need for interval-valued fuzzy sets: a brief reminder. In the traditional fuzzy logic, an expert describes his or her degree of confidence in different statements by a number from the interval $[0, 1]$. In particular, for statements like "x is small" corresponding to different values x, the corresponding degree $\mu(x)$ form a *membership function* describing the imprecise (fuzzy) concept like "small"; see, e.g., [1, 6].

In many practical situations, experts are not comfortable describing their degree of confidence by an exact number; they feel more comfortable describing their degree of confidence by an interval – e.g., by an interval $[0.7, 0.8]$. In particular, for statements like "x is small", the corresponding interval-valued degrees of confidence $\left[\underline{\mu}(x), \overline{\mu}(x)\right]$ form an *interval-valued* membership function.

The intuitive meaning of this membership function is that in principle, we can have many different number-valued membership functions $\mu(x)$ as long as $\mu(x) \in \left[\underline{\mu}(x), \overline{\mu}(x)\right]$ for every x.

Another case when an interval-valued membership function naturally appears is when we ask several experts. For the same value x, different experts give, in general, different degrees of confidence $\mu_1(x), \ldots, \mu_n(x)$. When experts are equally good, there is no reason to select one of these values, it make more sense to consider the interval $\left[\min\limits_i \mu_i(x), \max\limits_i \mu_i(x)\right]$ spanned by these values. This smallest interval containing the values $\mu_1(x), \ldots, \mu_n(x)$ is also known as the *interval hull* of the corresponding finite set $\{\mu_1(x), \ldots, \mu_n(x)\}$.

Need for multi-interval-valued fuzzy sets. Once each expert provides his or her degree $\mu_i(x)$ or interval-valued degree $\left[\underline{\mu}_i(x), \overline{\mu}_i(x)\right]$, then, instead of taking the interval hull of all these degrees, we can get a more adequate description of the experts' opinions if we simply take the *union* of these values and intervals. Such unions are known as *multi-intervals.* If for each x, the experts' degrees of confidence in the corresponding statement "x is small" is described by a multi-interval $M(x)$, then we get a multi-interval-valued membership function $M(x)$; see, e.g., [7].

Centroid defuzzification for regular fuzzy sets. In control (or, more generally, decision) applications, when for each possible value x of control, we know the degree $\mu(x)$ to which this value is reasonable, we then need to decide which control value c to apply.

In fuzzy applications, we usually select the value c for which the weighted mean square deviation from this value is the smallest possible:

$$\int\limits_{\underline{L}}^{\overline{L}} \mu(x) \cdot (x-c)^2\,dx \to \min_c,$$

where $\left[\underline{L}, \overline{L}\right]$ is the range of possible values of x. Differentiating this objective function with respect to the unknown c and equating the derivative to 0, we conclude that

$$c(\mu) = \frac{\int\limits_{\underline{L}}^{\overline{L}} x \cdot \mu(x)\,dx}{\int\limits_{\underline{L}}^{\overline{L}} \mu(x)\,dx}.$$

This formula is known as *centroid defuzzification.*

Centroid defuzzification for interval-valued fuzzy sets. As we have mentioned, an interval-values fuzzy set $\left[\underline{\mu}(x), \overline{\mu}(x)\right]$ means that many different membership functions $\mu(x) \in \left[\underline{\mu}(x), \overline{\mu}(x)\right]$ are possible. For different possible membership functions $\mu(x)$, in general, we have different defuzzification results $c(\mu(x))$. It is therefore reasonable to find the set of all possible value of these results:

$$\{c(\mu) : \underline{\mu}(x) \le \mu(x) \le \overline{\mu}(x) \text{ for all } x\}.$$

It is known (see, e.g., [3–5]) that this range is always an interval $[\underline{c}, \overline{c}]$, where

$$\underline{c} = \frac{\int\limits_{\underline{L}}^{x_-} x \cdot \overline{\mu}(x)\,dx + \int\limits_{x_-}^{\overline{L}} x \cdot \underline{\mu}(x)\,dx}{\int\limits_{\underline{L}}^{x_-} \overline{\mu}(x)\,dx + \int\limits_{x_-}^{\overline{L}} \underline{\mu}(x)\,dx}$$

and

$$\overline{c} = \frac{\int\limits_{\underline{L}}^{x_+} x \cdot \underline{\mu}(x)\,dx + \int\limits_{x_+}^{\overline{L}} x \cdot \overline{\mu}(x)\,dx}{\int\limits_{\underline{L}}^{x_+} \underline{\mu}(x)\,dx + \int\limits_{x_+}^{\overline{L}} \overline{\mu}(x)\,dx}$$

for appropriate values x^- and x^+. These formulas underlie the known algorithms for computing the range $[\underline{c}, \overline{c}]$.

Formulation of the problem. Now, we are ready to formulate our problem. What if instead of an interval-values fuzzy set, we have a multi-interval-valued fuzzy set $M(x)$? What will then be the set

$$\{c(\mu) : \mu(x) \in M(x) \text{ for all } x\}?$$

2 Analysis of the Problem and the Main Result

Discussion. A multi-set is a union of finitely many one-points sets and (closed) intervals. Each of the united sets is closed, thus their union $M(x)$ is closed. In general, we will consider functions that assign, to each value x, a closed set $M(x) \subseteq [0, 1]$; see, e.g., [7].

Each such closed set contains its own infimum $\underline{M}(x) \stackrel{\text{def}}{=} \inf M(x)$ and supremum $\overline{M}(x) \stackrel{\text{def}}{=} \sup M(x)$. The interval hall of the set $M(x)$ is the interval $\left[\underline{M}(x), \overline{M}(x)\right]$.

We assume that this function $M(x)$ is defined for all the values x from some interval $\left[\underline{L}, \overline{L}\right]$. We also assume that the lower and upper bounds $\underline{M}(x)$ and $\overline{M}(x)$

are measurable functions. It turns out that under these conditions, the centroid defuzzification of the set-valued membership function $M(x)$ is equivalent to the centroid defuzzification of its interval hull $\left[\underline{M}(x), \overline{M}(x)\right]$. Let us formulate this result in precise terms.

Definition. *By a* set-valued membership function*, we mean a function M that assigns, to each real number from some interval $\left[\underline{L}, \overline{L}\right]$, a closed set $M(x) \subseteq [0,1]$ for which the functions*

$$\underline{M}(x) = \inf M(x) \text{ and } \overline{M}(x) = \sup M(x)$$

are measurable.

Proposition. *For the centroid defuzzification functional $c(\mu)$, we have*

$$\{c(\mu) : \mu(x) \in M(x) \text{ for all } x\} = \{c(\mu) : \mu(x) \in [\inf M(x), \sup M(x)] \text{ for all } x\}.$$

Comment. Thus, the result of applying centroid defuzzification to the original set-valued fuzzy set $M(x)$ is equivalent to applying the same centroid defuzzification to its interval hull $\left[\underline{M}(x), \overline{M}(x)\right]$.

Proof.

1°. Let us first prove that out of all possible functions $\mu(x) \in M(x)$, the smallest and the largest possible values of $\mu(x)$ are attained when for each x, the value $\mu(x)$ is equal to either $\underline{M}(x)$ or to $\overline{M}(x)$.

It is sufficient to prove this result in the discrete case, when instead of the whole interval $\left[\underline{L}, \overline{L}\right]$, we have finitely many value $x_1, \ldots, x_n$: e.g., the values $x_i = \underline{L} + h \cdot (i-1)$, where $h = \dfrac{\overline{L} - \underline{L}}{n-1}$; the general case can be obtained when we take $n \to \infty$. In this discrete cases, instead of the whole membership function $\mu(x)$, we have n values $\mu(x_n)$, and the centroid defuzzification takes the form

$$c = \frac{\sum\limits_{i=1}^{n} x_i \cdot \mu(x_i) \cdot h}{\sum\limits_{i=1}^{n} \mu(x_i) \cdot h}.$$

Dividing both the numerator and the denominator of this expression by the common factor h, we get a simplified expression

$$c = \frac{\sum\limits_{i=1}^{n} x_i \cdot \mu(x_i)}{\sum\limits_{i=1}^{n} \mu(x_i)}.$$

Let us show that for each j, this expression is either monotonically increasing or monotonically decreasing as a function of $\mu(x_j)$. A monotonic function attains its maximum and its minimum on an interval on the endpoints of this interval, thus, the minimum and maximum are attained when either $\mu(x_j) = \underline{M}(x_j)$ or $\mu(x_j) = \overline{M}(x_j)$.

Let us prove the desired monotonicity. Indeed, the above expression for c can be described in the following equivalent form:

$$c = \frac{\sum\limits_{i \neq j} x_i \cdot \mu(x_i) + x_j \cdot \mu(x_j)}{\sum\limits_{i \neq j} \mu(x_i) + \mu(x_j)}.$$

If we subtract x_j from the right-hand side (and bring the difference to the common denominator) and then add x_j to the result, we get the following equivalent expression:

$$c = x_j + \frac{\sum\limits_{i \neq j} (x_i - x_j) \cdot \mu(x_i)}{\sum\limits_{i \neq j} \mu(x_i) + \mu(x_j)}.$$

The denominator is an increasing function of $\mu(x_i)$, and the numerator does not depend on $\mu(x_i)$ at all. Thus:

- if the numerator is positive, the expression is a decreasing function of $\mu(x_i)$, and
- if the numerator is negative, then the expression is an increasing function of $\mu(x_i)$.

The statement is proven.

2°. We have shown that the maximum and minimum of $c(\mu)$ – when for each x, we have $\mu(x) \in M(x)$ – is equal either to the smallest possible value $\underline{M}(x)$ or to the largest possible value $\overline{M}(x)$. Thus, the maximum and minimum of $c(\mu)$ over all $\mu(x) \in M(x)$ are equal to, correspondingly, the maximum and the minimum of $c(\mu)$ over all $\mu(x) \in \{\underline{M}(x), \overline{M}(x)\}$:

$$\overline{c} = \max\{c(\mu) : \mu(x) \in M(x) \text{ for all } x\} =$$

$$\max\left\{c(\mu) : \mu(x) \in \left\{\underline{M}(x), \overline{M}(x)\right\} \text{ for all } x\right\}$$

and

$$\underline{c} = \min\{c(\mu) : \mu(x) \in M(x) \text{ for all } x\} =$$

$$\min\left\{c(\mu) : \mu(x) \in \left\{\underline{M}(x), \overline{M}(x)\right\} \text{ for all } x\right\}.$$

As we have mentioned earlier, a similar property holds for interval-valued fuzzy sets, when instead of the restriction $\mu(x) \in M(x)$, we impose an interval restriction $\mu(x) \in [\underline{M}(x), \overline{M}(x)]$: here also,

$$\max\left\{c(\mu) : \mu(x) \in \left[\underline{M}(x), \overline{M}(x)\right] \text{ for all } x\right\} =$$

$$\max\left\{c(\mu) : \mu(x) \in \left\{\underline{M}(x), \overline{M}(x)\right\} \text{ for all } x\right\}$$

and

$$\min\left\{c(\mu) : \mu(x) \in \left[\underline{M}(x), \overline{M}(x)\right] \text{ for all } x\right\} =$$
$$\min\left\{c(\mu) : \mu(x) \in \left\{\underline{M}(x), \overline{M}(x)\right\} \text{ for all } x\right\}.$$

Thus, the maximum and minimum of $c(\mu)$ under the *set* condition $\mu(x) \in M(x)$ are equal to the maximum and minimum of $c(\mu)$ under the *interval* condition $\mu(x) \in \left[\underline{M}(x), \overline{M}(x)\right]$.

So, to complete our proof, we need to show that in both cases, every real number in between $\underline{c}$ and $\overline{c}$ belongs to the desired range, i.e., has the form $c(\mu)$ for an appropriate membership function $\mu(x)$, i.e., a membership function for which we have either $\mu(x) \in M(x)$ (in the set case) or $\mu(x) \in [\underline{M}(x), \overline{M}(x)]$ (in the interval case).

We will show that for every $c \in [\underline{c}, \overline{c}]$, we can select a function $\mu(x)$ for which $\mu(x) \in \{\underline{M}(x), \overline{M}(x)\}$ – this would guarantee both that $\mu(x) \in M(x)$ and that $\mu(x) \in [\underline{M}(x), \overline{M}(x)]$.

To prove the existence of such a function, let us start with the functions $\mu_-(x) \in \{\underline{M}(x), \overline{M}(x)\}$ and $\mu_+(x) \in \{\underline{M}(x), \overline{M}(x)\}$ for which $c(\mu_-) = \underline{c}$ and $c(\mu_+) = \overline{c}$. For each value $\ell \in [\,\underline{L}, \overline{L}\,]$, we can now consider an auxiliary function $\mu_\ell(x)$ which is:

- equal to $\mu_+(x)$ for $x \le \ell$ and
- equal to $\mu_-(x)$ for $x > \ell$.

For each x, the value of $\mu_\ell(x)$ is equal to either the value $\mu_-(x)$ or to the value $\mu_+(x)$. Since both of these values are from the set $\{\underline{M}(x), \overline{M}(x)\}$, the value $\mu_\ell(x)$ also belongs to this set for all x.

Since we assumed that the functions $\underline{M}(x)$ and $\overline{M}(x)$ are measurable, we can conclude that the value $c(\mu_\ell)$ is a continuous function of ℓ.

When $\ell = \underline{L}$, the function $\mu_\ell(x)$ coincides with $\mu_-(x)$, and for $\ell = \overline{L}$, it coincides with $\mu_+(x)$. Thus, as ℓ changes from $\underline{L}$ to $\overline{L}$, the value of $c(\mu_\ell)$ continuously changes from $c(\mu_-) = \underline{c}$ to $c(\mu_+) = \overline{c}$. A continuous function attains all intermediate values, so for each $c \in [\underline{c}, \overline{c}]$, there indeed exists a value ℓ for which $c(\mu_\ell) = c$, for the corresponding function $\mu_\ell(x) \in \{\underline{M}(x), \overline{M}(x)\}$.

The statement is proven, and so is the proposition.

3 From Set-Valued to General Type-2 Fuzzy Sets

Type-2 fuzzy sets: reminder. Instead of considering, for each x, a crisp set $M(x)$ of possible values of the degree of confidence $\mu(x)$, it makes sense to consider a more general case, when this set of possible values of the degree is fuzzy. Such situations are known as *type-2 fuzzy sets*; see, e.g., [4, 5].

In precise terms, for each x and for each real number $\mu \in [0, 1]$, instead of deciding whether this number is a possible value of the degree or not, we now have a *degree* $d(\mu, x)$ describing to what extent the number μ is a possible expert's degree of confidence that x satisfies the given property (e.g., "is small").

Centroid defuzzification: general type-2 case. We have a functional $c(\mu)$ defined for crisp function $\mu(x)$. In fuzzy techniques, a natural way to extend this functional to fuzzy-valued membership functions – i.e., to type-2 fuzzy sets – is to use Zadeh's extension principle.

It is known that this principle can be equivalently described in terms of α-cut: for any function $y = f(x_1, \ldots)$ and for fuzzy sets $X_1, \ldots$, the α-cut $Y(\alpha) \stackrel{\text{def}}{=} \{y : \mu_Y(y) \geq \alpha\}$ of the result $Y = f(X_1, \ldots)$ of the result of applying the function f to fuzzy sets $X_1, \ldots$ is equal to the range of the function on the alpha-cuts $X_i(\alpha) = \{x_i : \mu_i(x_i) \geq \alpha\}$ of the inputs X_i:

$$Y(\alpha) = f(X_1(\alpha), \ldots) = \{f(x_1, \ldots) : x_1 \in X_1(\alpha), \ldots\}.$$

In particular, for the centroid defuzzification, we start with a function $c(\mu)$ that depends on infinitely many real-valued inputs $\mu(x)$. For type-2 fuzzy sets, the inputs $\mu(x)$ are also fuzzy. Thus, the result of a centroid defuzzification is also a fuzzy set C. the α-cut $C(\alpha)$ of this fuzzy set C of defuzzification results is equal to the range of the values $c(\mu)$ under the condition that for all x, we have $\mu(x) \in M_x(\alpha) = \{\mu : d(\mu, x) \geq \alpha\}$.

Consequence of our main result: centroid defuzzification of a general type-2 fuzzy set can be reduced to the convex case. Our main result states that for each set-valued function $M(x)$, the range of the centroid defuzzification is equal to the range of its interval hull $\left[\,\underline{M}(x), \overline{M}(x)\,\right]$.

Thus, for each type-2 membership function $d(\mu, x)$, the range $C(\alpha)$ is equal to the range computed based on the interval hull $[\inf M_x(\alpha), \sup M_x(\alpha)]$ of the set $M_x(\alpha)$.

In other words, the result C of applying the centroid defuzzification $c(\mu)$ to the general type-2 fuzzy set is equal to the result of applying $c(\alpha)$ to an auxiliary fuzzy set in which each α-cut is a (convex) interval $[\inf M_x(\alpha), \sup M_x(\alpha)]$. Thus, *centroid defuzzification of a general type-2 fuzzy set can indeed be reduced to the convex case* – the case for which there exist efficient algorithms; see, e.g., [2].

Acknowledgments

We acknowledge the partial support of the Center of Excellence in Econometrics, Faculty of Economics, Chiang Mai University, Thailand. This work was also supported in part by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and DUE-0926721, and by an award "UTEP and Prudential Actuarial Science Academy and Pipeline Initiative" from Prudential Foundation.

The authors are thankful to all the participants of the 2016 IEEE World Congress on Computational Intelligence WCCI'2016 (Vancouver, Canada, July 24–29, 2016) for valuable discussions.

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