Metric Spaces Under Interval Uncertainty: Towards an Adequate Definition

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Abstract. In many practical situations, we only know the bounds on the distances. A natural question is: knowing these bounds, can we check whether there exists a metric whose distance always lie within these bounds – or such a metric is not possible and thus, the bounds are inconsistent. In this paper, we provide an answer to this question. We also describe possible applications of this result to a description of opposite notions in commonsense reasoning.

1 An AI Problem and the Resulting Mathematical Problem

Starting point: commonsense negation vs. formal negation. Negation and opposites are an important part of our reasoning. Thus, to better understand human reasoning, it is desirable to analyze how we use negation.

The standard way to describe negation is to use mathematical logic. In mathematical logic, negation has a very precise meaning: a negation $\neg S$ of a statement S is true if and only if the statement S is false. This formal logical notion of a negation corresponds to the notion of a complement to a set: a complement $-S$ is the set of all the objects that do not belong to the original set S.

Similarly, in fuzzy logic (see, e.g., [7, 8, 10]):

- once we have a fuzzy set, i.e., a function μ_P that assigns, to each object x, a degree $\mu(x)$ to which this object satisfies a given imprecise property P (e.g., is small),
- then the negation is usually defined as a membership function

$$\mu_{\neg P} x = 1 - \mu_P(x).$$

The logical notion of negation corresponds to the intuitive idea of an opposite. However, in contrast to the formal negation – which is uniquely determined by the original concept – there can be many different opposite to a given notion, depending on a context. For example, depending on a context, the opposite to a man is either a boy or a woman (see, e.g., [9], where an interesting formalism is developed for describing opposites).

Why there may be several different opposites to the same notion: a natural explanation. In our opinion, the existence of several different opposites has a simple explanation: when we reason, we try our best to use simple, basic concepts. A formal negation of the notion of a man is *not* an intuitively simple concept. So, instead of using this complicated concept, we select one of the basic concepts: namely, the one which is the closest to the original negation.

Depending on the context, we may have different metrics, and thus, different concepts are the closest to the original negation.

Beyond negation. A similar idea can be applied to other logical connectives such as "and" and "or": instead of the original formal intersection or union, we the basic notion which is the closest to the corresponding formal result.

Comment. The very fact that, depending on the context, "and" and "or" may have different meanings, is well known. For example, one of the main motivations behind linear logic (see, e.g., [5, 6]) was to formally explain the difference between different commonsense meanings of "and".

Let us formalize this idea. Let us describe this idea in precise terms. The distance is usually described as a *metric*, i.e., as a function $d : X \times X \to [0, \infty)$ that assigns, to every two objects a and b from the universal set X, a non-negative number $d(a, b)$ with the following properties:

- first, $d(a, b) = 0$ if and only if $a = b$;
- second, $d(a, b) = d(b, a)$, and
- finally, we must have the triangle inequality $d(a, c) \le d(a, b) + d(b, c)$.

In the case of negation, we have a list of basic notions $A_1, \ldots, A_n$, and we have their negations $\neg A_1, \ldots, \neg A_n$. For each concept A_i, we need to select the concept A_j which is, in a given metric d, the closest to $\neg A_i$, i.e., for which

$$d(A_j, \neg A_i) = \min_k d(A_k, \neg A_i).$$

Similarly, to describe the concept corresponding to $A_i \,\&\, A_j$, we need to select a concept A_k which is the closest to the conjunction $A_i \,\&\, A_j$:

$$d(A_k, A_i \,\&\, A_j) = \min_\ell d(A_\ell, A_i \,\&\, A_j).$$

The resulting mathematical problem: first approximation. In principle, in the case of n concepts and their negations, we have $2n$ objects, thus, we can have distances $d(A_i, A_j)$, $d(A_i, \neg A_j)$, and $d(\neg A_i, \neg A_j)$. To make the above selection of the opposite A_j to A_i (or of the corresponding disjunction), we do not need to know the distances $d(A_i, A_j)$ and $d(\neg A_i, \neg A_j)$, we only need to know the distances $d(A_i, \neg A_j)$.

While we *do not need* to know the values $d(A_i, A_j)$ and $d(\neg A_i, \neg A_j)$, to analyze all possible situations, we *do need* to make sure that the distances $d(A_i, \neg A_j)$ are such that for *some* values $d(A_i, A_j)$ and $d(\neg A_i, \neg A_j)$, we get the triangle inequality and all the properties of the metric.

Similarly, to describe a commonsense "and", we may not need to know the distances $d(A_i, A_j)$ and $d(A_i \,\&\, A_j, A_k \,\&\, A_\ell)$, but we must make sure that there exist some values that, combined with the known values $d(A_i, A_j \,\&\, A_k)$, form a metric.

The mathematical problem: towards a final formulation. The above description assumes that a person can give us the exact number $d(A_i, \neg A_j)$ describing the similarity between the basic concept A_i and the negation $\neg A_j$. In reality, people can usually only make approximate judgments about their opinions. Thus, at best, a person will provide us with some *bounds* $\underline{d}(A_i, \neg A_j)$ and $\overline{d}(A_i, \neg A_j)$ so that the actual (unknown) distance lies somewhere in the interval

$$[\underline{d}(A_i, \neg A_j), \overline{d}(A_i, \neg A_j)].$$

Thus, to each pair $(a, b) = (A_i, \neg A_j)$, instead of a real number $d(a, b)$, we assign an interval $[\underline{d}(a, b), \overline{d}(a, b)]$ of possible values. We are then facing the same problem: when does there exist a metric $d(a, b)$ for which, for all these pairs (a, b), we have $d(a, b) \in [\underline{d}(a, b), \overline{d}(a, b)]$?

Since we allow intervals anyway, we can describe the fact that we know nothing about the distances such $d(A_i, A_j)$ by assigning to each such pair $(a, b) = (A_i, A_j)$, an infinite interval $[0, \infty)$. Thus, we arrive at the following problem.

Resulting mathematical problem. We have a final set X. For every two elements a and b from this set, we have an interval $[\underline{d}(a, b), \overline{d}(a, b)]$, where the upper bound $\overline{d}(a, b)$ may be infinite.

We would like to find the conditions on these intervals which are equivalent to the existence of a metric $d(a, b)$ for which $d(a, b) \in [\underline{d}(a, b), \overline{d}(a, b)]$ for all a and b.

An important particular case. An important particular case of this problem is when – like in case of negation or disjunction – the set X consists of two disjoint subsets X^+ and X^-, so that we only know the distances between the elements of X^+ and X^-.

In the negation example, X^+ is the set of all basic notions A_i, and X^- is the set of all negations $\neg A_i$. In the disjunction example, X^+ is the set of all basic notions, while X^- is the set of all possible formal disjunctions $A_i \,\&\, A_j$, etc.

2 Towards Solving the Mathematical Problem: How Are Interval-Valued Metric Spaces Defined Now

Current definition: motivations. The need to extend metric spaces to the case of interval uncertainty has been recognized for a few decades already. There exist natural interval-valued extensions of metric spaces; see, e.g., [1–4]. Before we give the corresponding definition, let us first explain the motivations behind this definition.

The main property of a metric $d(a, b)$ is that it must satisfy the triangle inequality $d(a, c) \le d(a, b) + d(b, c)$ for all a, b, and c.

In the case of interval uncertainty, we do not know the exact values $d(a, b)$, $d(b, c)$, and $d(a, c)$, we only know the intervals $[\underline{d}(a, b), \overline{d}(a, b)]$, $[\underline{d}(b, c), \overline{d}(b, c)]$, and $[\underline{d}(a, c), \overline{d}(a, c)]$ that contain these values. It is therefore reasonable to require that for all a, b, and c, the corresponding three intervals are selected in such a way that the triangle inequality is satisfied for some values from the corresponding intervals.

This condition is easy to describe. When $d(a, b) \in [\underline{d}(a, b), \overline{d}(a, b)]$ and $d(b, c) \in [\underline{d}(b, c), \overline{d}(b, c)]$, then the possible values of the sum $d(a, b) + d(b, c)$ form the interval $[\underline{d}(a, b) + \underline{d}(b, c), \overline{d}(a, b) + \overline{d}(b, c)]$. A value can be smaller than one of the values from this interval if it is smaller than its upper bound $\overline{d}(a, b) + \overline{d}(b, c)$.

Thus, the triangle inequality is satisfied if at least one value $d(a, c)$ from the interval $[\underline{d}(a, c), \overline{d}(a, c)]$ is smaller than or equal to the sum $\overline{d}(a, b) + \overline{d}(b, c)$. Of course:

- if a value $d(a, c)$ from the interval $[\underline{d}(a, c), \overline{d}(a, c)]$ is smaller than equal that the sum, then the lower endpoint $\underline{d}(a, c) \le d(a, c)$ is also smaller than or equal to the sum;
- vice versa, if the lower endpoint $\underline{d}(a, c)$ is smaller than or equal to the sum, then, since this endpoint belongs to the interval $[\underline{d}(a, c), \overline{d}(a, c)]$, we have a value $d(a, c) \in [\underline{d}(a, c), \overline{d}(a, c)]$ which is smaller than or equal to the sum.

Thus, the existence of the values $d(a, b) \in [\underline{d}(a, b), \overline{d}(a, b)]$, $d(b, c) \in [\underline{d}(b, c), \overline{d}(b, c)]$, and $d(a, c) \in [\underline{d}(a, c), \overline{d}(a, c)]$ for which the triangle inequality is satisfied is equivalent to the following inequality

$$\underline{d}(a, c) \le \overline{d}(a, b) + \overline{d}(b, c).$$

And this is how interval-valued metric spaces are defined now: that the above inequality holds for all possible a, b, and c.

Problem with the current definition. While every interval-valued metric that contains the actual metric $d(a, b)$ must satisfy the above inequality, it turns out that this inequality is not sufficient to guarantee that there is a metric inside the corresponding intervals.

Indeed, let us consider the case when $d(A_i, \neg A_j) = 1$ for all i and j except for $d(A_1, \neg A_1) = 4$, and when for $d(A_i, A_j)$ and $d(\neg A_i, \neg A_j)$ we only know that these values are in the infinite interval $[0\infty)$.

In this case, the above inequality is trivially satisfied, since there are no a, b, and c for which for all three distances $d(a, b)$, $d(b, c)$, and $d(a, c)$, there will be non-trivial interval. Indeed:

- If we have a finite intervals for $d(a, b)$, this means that a and b belong to different subsets X^+ and X^-.

– Similarly, if $d(b,c)$ is finite, this means that b and c belong to different subsets.
– Thus, a and c belong to the same subset – and thus, we only have an infinite bound for $d(a,c)$.

On the other hand, if we have a metric $d(a,b) \in [\underline{d}(a,b), \overline{d}(a,b)]$, then from the triangle inequality, we would be able to conclude that

$$d(A_1, \neg A_1) \le d(A_1, \neg A_2) + d(A_2, \neg A_2) + d(A_2, \neg A_1).$$

Here, the right-hand side is 3, but $d(A_1, \neg A_1) = 4 > 3$.

So, the usual definition of an interval-valued metric space is satisfied, but still no metric is possible. Thus, to solve our problem, we need to come up with a more adequate definition.

3 Solving the Mathematical Problem: Definition and the Main Result

Definition 1. *By an* interval-valued metric *on a set X, we mean a mapping that assigns, to each pair of elements a, b from the set X, an interval $[\underline{d}(a,b), \overline{d}(a,b)]$ with $\underline{d}(a,b)$ that satisfies the following properties:*

– *first, $d(a,a) = [0,0]$ and $\overline{d}(a,b) > 0$ for $a \ne b$;*
– *second, $\underline{d}(a,b) = \underline{d}(b,a)$ and $\overline{d}(a,b) = \overline{d}(a,b)$, and*
– *finally, for every finite chain $a_1, a_2, \ldots, a_m$, we have*

$$\underline{d}(a_1, a_m) \le \overline{d}(a_1, a_2) + \overline{d}(a_2, a_3) + \ldots + \overline{d}(a_{m-1}, a_m).$$

Discussion. It is easy to check that if we have an interval-valued metric that contains the actual metric, then the above version of triangle inequality must be satisfied. It turns our that, vice versa, once this inequality is satisfied, there exists a metric contained in all these intervals.

Proposition 1. *For every interval-valued metric space X with an interval metric $[\underline{d}(a,b), \overline{d}(a,b)]$, there exists a metric $d(a,b)$ for which $d(a,b) \in [\underline{d}(a,b), \overline{d}(a,b)]$ for all a and b.*

Proof.

1°. Let us first consider the case when every two elements $a, b \in X$ can be connected by a chain $a = a_1, a_2, \ldots, a_{m-1}, a_m = b$ for which $\overline{d}(a_i, a_{i+1}) < +\infty$ for every i.

In this case, let us take

$$d(a,b) = \inf \left\{ \sum_{i=1}^{m-1} \overline{d}(a_i, a_{i+1}) \right\},$$

where the infimum is taken over all the chains connecting a and b.

2°. Let us prove that $\underline{d}(a,b) \le d(a,b) \le \overline{d}(a,b)$ for all a and b.

Indeed, by our version of the triangle inequality, $\underline{d}(a,b)$ is smaller than or equal than each of the sums $\sum\limits_{i=1}^{m-1} \overline{d}(a_i,a_{i+1})$. Thus, it is smaller than or equal to the smallest of these sums, i.e., indeed, $\underline{d}(a,b) \le d(a,b)$.

On the other hand, the chain $a_1 = a$, $a_2 = b$ is one of the possible chains connecting a ad b. For this chain, the sum $\sum\limits_{i=1}^{m-1} \overline{d}(a_i,a_{i+1})$ is simply equal to $\overline{d}(a,b)$. Since $d(a,b)$ is the smallest of these sums, we thus conclude that

$$d(a,b) \le \overline{d}(a,b).$$

3°. Let us now prove that the function $d(a,b)$ satisfies the triangle inequality, i.e., that $d(a,c) \le d(a,b) + d(b,c)$. Indeed, let $a = a_1, a_2, \ldots, a_m = b$ be a chain connecting a and b for which the sum is the smallest (and is equal to $d(a,b)$). Let $b = b_1, \ldots, b_p = c$ be the chain connecting b and c for which the sum is the smallest – and is equal to $d(b,c)$. Then, for the combined chain $a_1, a_2, \ldots, a_m = b_1, b_2, \ldots, b_p$ the sum is equal to the sum of the sums corresponding to the two chains, i.e., to $d(a,b)+d(b,c)$. Since $d(a,v)$ is the smallest over all possible chains – not necessarily passing through b – we thus have $d(a,c) \le d(a,b) + d(b,c)$.

4°. Let us now consider the general case, when the relation

"a and b can be connected by a chain in which $d(a_i,a_{i+1}) < +\infty$"

divides the original finite set X into several equivalence classes.

Within each equivalence class, the above formula for $d(a,b)$ describes a metric. Let us select a point within each equivalent class. For each element $a \in X$, let us describe the point selected for the corresponding equivalent class by a_0.

For elements a and b that belong to different equivalent classes, let us defined the distance $d(a,b)$ as $d(a,b) \stackrel{\text{def}}{=} d(a,a_0) + d(b,b_0) + 1$. Let us show that thus extended distance satisfies the triangle inequality $d(a,c) \le d(a,b) + d(b,c)$ for every triple (a,b,c).

To prove this inequality, let us list all possible cases.

- First, we need to consider the case when all three elements a, b, and c belong to the same equivalence class. In this case, the triangle inequality follows from the fact that on each equivalence class, $d(a,b)$ is a metric.
- The next case when two of the elements belong to the same equivalence class, while the third element belongs to a different equivalence class. We have to consider two subcases of this case:
  - the subcase when a and c belong to the same equivalence class, and
  - the subcase when a and b belong to the same equivalence class (the subcase when b and c belong to the same equivalence class is similar).
- Finally, we need to consider the case when all three elements a, b, and c belong to different equivalence classes.

Let us consider these cases one by one.

4.1°. Let us first consider the case when a and c belong to the same equivalence class (so $a_0 = c_0$), and b belongs to a different equivalent class.

In this case, the desired inequality has the form

$$d(a, c) \le d(a, a_0) + d(b, b_0) + 1 + d(b, b_0) + d(c, a_0) + 1.$$

This is indeed true: from the triangle inequality for the equivalence class containing a and c, we get $d(a, c) \le d(a, a_0) + d(c, a_0)$. Since $d(b, b_0) \ge 0$, we get the desired inequality

$$d(a, c) \le d(a, a_0) + d(c, a_0) \le d(a, a_0) + d(b, b_0) + 1 + d(b, b_0) + d(c, a_0) + 1.$$

4.2°. Let us now consider the case when a and b belong to the same equivalence class (so $a_0 = b_0$), and c belongs to a different equivalence class.

In this case, the desired inequality takes the form

$$d(a, a_0) + d(c, c_0) + 1 \le d(a, b) + d(b, a_0) + d(c, c_0) + 1.$$

Indeed, for a, c and a_0, we have the triangle inequality $d(a, a_0) \le d(a, b) + d(b, a_0)$. By adding $d(c, c_0) + 1$ to both sides, we get the desired inequality.

4.3°. Finally, let us consider the case when all three elements a, b, and c belong to different equivalence classes. In this case, the desired inequality has the form

$$d(a, a_0) + d(c, c_0) + 1 \le d(a, a_0) + d(b, b_0) + 1 + d(b, b_0) + d(c, c_0) + 1.$$

Indeed, the right-hand side of this inequality is obtained from the left-hand side by adding the sum $2d(b, b_0) + 1$. Since $d(b, b_0) \ge 0$, the added sum is always positive, so the inequality is indeed always true.

We have considered all possible cases, so the proposition is proven.

4 Auxiliary Result: What About the Original Case When We Have Two Disjoint Subsets

Now that we have proven a general result, let us consider the case when the step X is divided into two disjoint sets.

Proposition 2. *Let a set X be a union of two disjoint subsets X^+ and X^-. Assume that we know the values $d(a, b) \ge 0$ for every pair (a, b) in which $a \in X^+$ and $b \in X^-$. Then, the following two conditions are equivalent to each other:*

- *there exist a metric $d(a, b)$ whose restriction to pairs $(a \in X^+, b \in X^-)$ coincides withe the given values, and*
- *for every $a, a' \in X^+$ and $b, b' \in X^-$, we have*

$$d(a, b') \le d(a, a') + d(b, a') + d(b, b').$$

Proof. If there is a metric extending given values, then the above inequality follows from the triangle inequality: $d(a,b') \le d(a,a') + d(b,b')$ and $d(b,b') \le d(b,a') + d(b,b')$, hence indeed $d(a,b') \le d(a,a') + d(b,a') + d(b,b')$.

Vice versa, let us assume that the above equality is always satisfied. Let us then show that the given values $a(a,b)$ satisfy the inequality from Proposition 1. This case is a particular case of the general interval-valued metric space, when we have $\underline{d}(a,b) = \overline{d}(a,b) = d(a,b)$ when a and b belong to different subsets and $\underline{d}(a,b) = 0$ and $\overline{d}(a,b)$ when a and b belong to different subsets.

If a and b belong to the same subset, then $\underline{d}(a,b) = 0$ and the condition from Proposition 1 is trivially satisfied.

If a and b belong to different subsets, this means that $a = a_1$ and a_2 belong to different subsets, a_2 and a_3 belong to different subsets, etc. In other words, a_1, a_3, ... belong to one subset, while a_2, a_4, etc. belong to the opposite subset. The above inequality implies that

$$d(a,b) = \underline{d}(a,b) \le \overline{d}(a,a_2) + \overline{d}(a_2,a_3) + \overline{d}(a_3,b) = d(a,a_2) + d(a_2,a_3) + d(a_3,b).$$

Similarly, $d(a_3,b) \le d(a_3,a_4) + d(a_4,a_5) + d(a_5,b)$, hence

$$d(a,b) \le d(a_1,a_2) + \ldots + d(a_5,b),$$

etc., so we get the desired inequality for chains for arbitrary length.

Since all the inequalities from Proposition 1 are satisfied, by Proposition 1, there exists the desired metric. The proposition is proven.

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