IV-я Международная научная конференция ФКН
«Математическое и компьютерное моделирование»,
Россия, Омск, ОмГУ, 11 ноября 2016 г. – С. 1–5. УДК XXX

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WHY UTILITY NON-LINEARLY DEPENDS ON MONEY: A COMMONSENSE EXPLANATION

Outline. Human decision making is based on the notion of utility. Empirical studies have shown that utility non-linearly depends on the money amount. In this paper, we provide a commonsense explanation of this empirical fact: namely, that without such non-linearity, we would not have a correct description of such a commonsense behavior as saving money for retirement.

Saving money for retirement: a simplified description of the problem. Let us consider a simplified version of this situation, when we only have two moments of the time: the current moment of time (when we earn money), and the future moment of time, in which we will not earn money.

Suppose that at the present moment, we earn the amount m. Out of this amount, we can save $s \le m$ and thus, spend the remaining amount $m - s$. The saved money is invested; as a result, with interest, in the future, we will have an increased amount $k \cdot s$, for some constant $k > 1$.

The question is how much money s we shall save, i.e., which amount $s \in [0, m]$ we should select.

How should we make this decision? According to the decision making theory, preferences and decisions by a rational decision maker are described by utilities of different alternatives; see, e.g., [3–5].

Let us briefly recall what is utility and how it is related to decision making.

Utility: a brief reminder. How can we describe human preferences? One possibility is to select two theoretically possible alternatives: one very bad A_-, much worse than any other alternative A that we will ever encounter ($A_- < A$), and another very good A_+, much better than anything that we will encounter in practice ($A < A_+$).

Then, for each number p from the interval $[0, 1]$, we can form a lottery $L(p)$ in which we get A_+ with probability p and A_- with the remaining probability $1 - p$.

Let us now consider an arbitrary alternative A which is in between A_- and A_+: $A_- < A < A_+$. When $p = 0$, we have $L(p) = A_- < A$; when $p = 1$, we have $L(p) = A_+ > A$. Thus, as we increase p from 0 to 1, there should be a threshold value p_0 at which A switches for being better than $L(p)$ to being worse than $L(p)$, i.e., for which, in this sense, $L(p_0)$ is "equivalent" to A: $A \sim L(p_0)$.

This threshold value p_0 is known as the *utility* of the alternative A. This value is usually denoted by $u(A)$, so that $A \sim L(u(A))$.

Utility of money. The utility depends on the alternative. In particular, for alternatives consisting of getting a monetary amount, the utility u depends on this amount a. Let us denote this dependence by $u = M(a)$.

In the savings situation, the current utility is thus equal to $u_c = M(m - s)$, and the future utility is equal to $u_f = M(k \cdot s)$.

Empirical fact: utility is a non-linear function of money. Empirical analysis shows that utility non-linearly depends on the money amount. The corresponding dependence is close to $M(a) = \sqrt{a}$; see, e.g., [1–2].

Why: a problem. The question is how can we explain this empirical fact. Such an explanation – based on the saving situation – is described in this paper.

Before we proceed with this explanation, we need to dip deeper into the relation between utility and decision making.

Expected utility: a reminder. Utility describes the desirability of each outcome. Our goal, however, is usually not to select an outcome, but rather to select an action. Usually, we are not 100% sure about the outcome of each action; each action can lead to different possible outcomes $A_1, \ldots, A_n$, with different probabilities $p_1, \ldots, p_n$. How do we describe desirability of such an action?

To describe this desirability, we can use the fact that each outcome A_i is equivalent to a lottery $L(u(A_i))$ in which we get A_+ with probability $u(A_i)$ and A_- with the remaining probability $1 - u(A_i)$. Thus, the corresponding action is equivalent to a composite lottery in which we first select one of the outcomes A_i with the corresponding probability p_i, and then, depending on the selected A_i, select A_+ or A_- with the probabilities $u(A_i)$ and $1 - u(A_i)$.

In this composite lottery, we get either A_+ or A_-. The probability p of getting A_+ can be computed by using the formula of complete probability, as $p_1 \cdot u(A_1) + \ldots + p_n \cdot u(A_n)$. Thus, the action is equivalent to the lottery $L(p)$ with this probability p. By definition of utility, it means that the utility u of the corresponding action is equal to this probability p, i.e., that

$$u = p_1 \cdot u(A_1) + \ldots + p_n \cdot u(A_n).$$

In mathematical terms, the right-hand side is the expected value of the utility $u(A_i)$ of the outcomes. Thus, the utility of an action is equal to the expected value of the utility of outcomes.

How unique is utility. The above definition of utility depends on the selection of the alternatives A_- and A_+. One can check that if instead, we select a different pair of extreme alternatives $A'_- < A'_+$, then the resulting utility values $u'(A)$ are related to the original values $u(A)$ by a linear dependence:

$$u'(A) = a \cdot u(A) + b,$$

for some $a > 0$ and b.

Thus, utility is defined modulo an arbitrary increasing linear transformation. The numerical value of the utility depends on the choice of the two auxiliary alternatives A_- and A_+. Thus, it makes sense that the formulas involving utilities should not change if we simply re-scale the utilities by using a different pair of alternative utilities – i.e., by applying the appropriate linear re-scaling.

How to take into account future utility. Let us use the above invariance argument to describe how a person will make a savings decision, a decision that affects not only the current situation, but also the future one.

Let u_c be the utility of the current situation, and let u_f denote the utility of a future situation. In the savings case, $u_f \le u_c$.

Different possible outcomes can be described by different pairs (u_c, u_f). To describe preferences between outcomes, we need to assign, to each such pair, a utility value u that describes the preference of the outcome characterized by this pair. Thus, we need to describe a function $u(u_c, u_f)$ that combines the original values u_c and u_f into a single utility value.

For this function, the above requirement means that if we re-scale the utilities u_c and u_f, then the resulting utility u will be similarly re-scaled, i.e., that for every $a > 0$ and for every b, we have

$$u(a \cdot u_c + b, a \cdot u_f + b) = a \cdot u(u_c, u_f) + b.$$

In particular, for $a = 1$ and $b = -u_c$, this implies that $u(0, u_f - u_c) = u(u_c, u_f) - u_c$, i.e., that

$$u(u_c, u_f) = u_c + F(u_c - u_f),$$

where we denoted $F(x) \stackrel{\text{def}}{=} u(0, -x)$.

For this expression, scale-invariance ($b = 0$) implies that

$$a \cdot u_c + F(a \cdot (u_c - u_f)) = a \cdot (u_c + F(u_c - u_f)) = a \cdot u_c + a \cdot F(u_c - u_f),$$

i.e., that $F(a \cdot (u_c - u_f)) = a \cdot F(u_c - u_f)$. If we denote $y \stackrel{\text{def}}{=} u_c - u_f$, then the above equality implies that $F(a \cdot y) = a \cdot F(y)$. For $y = 1$, this implies that $F(a) = c \cdot a$ for some constant $c \stackrel{\text{def}}{=} F(1)$. Thus,

$$u(u_c, u_f) = u_c + F(u_c - u_f) = u_c + c \cdot (u_f - u_c) = (1 - c) \cdot u_c + c \cdot u_f.$$

So, the resulting utility u linearly depends on the utilities u_c and u_f.

For $u_c = M(m - s)$ and $u_f = M(k \cdot s)$, we thus get

$$u(s) = u(u_c(s), u_f(s)) = (1 - c) \cdot M(m - s) + c \cdot M(k \cdot s).$$

We select the saved amount s for which this utility value is the largest possible.

What if utility linearly depends on money amount? If $M(a)$ is a linear function of a, then u is a linear function of the saved amount s. A linear function attains its largest values on the endpoints. Thus, for a linear utility function, we end up with one of the the following two options:

- the first option is $s = 0$, when we do not save anything for retirement at all;
- the second option is when we save the largest possible amount, i.e., the amount for which $u_f = M(k \cdot s) = u_c = M(m - s)$; in other words, we make sure that our retirement income is 100% of our original income.

This is not how people save for retirement; so, we have an explanation for the non-linear dependence. In reality, people save *some* money for retirement, but *not* the maximal amount of money: the retirement income is usually smaller than the original income.

So, in the simplest case of saving for retirement, models is which utility linearly depends on money amount do not describe the usual human behavior – which means that the dependence of utility on money amount should be non-linear.

Thus, we get the desired commonsense explanation of the empirical non-linearity.

Acknowledgments. This work was supported in part by the National Science Foundation grants HRD-0734825, HRD-1242122, and DUE-0926721.

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