Can We Detect Crisp Sets Based Only on the Subsethood Ordering of Fuzzy Sets? Fuzzy Sets And/Or Crisp Sets Based on Subsethood of Interval-Valued Fuzzy Sets?

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Abstract Fuzzy sets are naturally ordered by the subsethood relation $A \subseteq B$. If we only know which set which fuzzy set is a subset of which – and have no access to the actual values of the corresponding membership functions – can we detect which fuzzy sets are crisp? In this paper, we show that this is indeed possible. We also show that if we start with interval-valued fuzzy sets, then we can similarly detect type-1 fuzzy sets and crisp sets.

1 Formulation of the Problem

Fuzzy sets: a brief reminder. A *fuzzy set* is usually defined as a function $\mu : U \to [0,1]$ from some set U (called *Universe of discourse*) to the interval $[0,1]$; see, e.g., [1, 2, 3]. This function is also known as a *membership function*.

A fuzzy set A with a membership function $\mu_A(x)$ is called a *subset* of a fuzzy set B with a membership function $\mu_B(x)$ if $\mu_A(x) \le \mu_B(x)$ for all x. The subsethood relation is an *order* in the sense that it is reflexive ($A \subseteq A$), asymmetric ($A \subseteq B$ and $B \subseteq A$ imply $A = B$), and transitive ($A \subseteq B$ and $B \subseteq C$ imply $A \subseteq C$).

Traditional (*crisp*) sets S can be viewed as particular cases of fuzzy sets, with their characteristic functions playing the role of membership functions: $\mu_S(x) = 1$ if $x \in S$ and $\mu_S(x) = 0$ if $x \notin S$.

A natural question: can we detect crisp sets based only on the subsethood ordering of fuzzy sets? If we have a class F of all fuzzy sets, and for each fuzzy

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set A and for each element $x \in U$, we know the value $\mu_A(x)$ of the corresponding membership function, then we can easily detect which of the fuzzy sets are crisp: a fuzzy set is crisp if for every $x \in U$, we have either $\mu_A(x) = 0$ or $\mu_A(x) = 1$.

Suppose now that we have a class F of all fuzzy sets with the subsethood ordering $A \subseteq B$ – but we have no access to the actual values of the corresponding membership functions. Based only on this ordering relation $A \subseteq B$, can we then detect crisp sets?

What if we only consider interval-valued fuzzy sets. A similar question can be asked if we consider interval-valued fuzzy sets, for which the value of the membership function is a subinterval of the interval $[0,1]$: $\mu(x) = [\underline{\mu}(x), \overline{\mu}(x)] \subseteq [0,1]$, and $A \subseteq B$ means that $\underline{\mu}_A(x) \le \underline{\mu}_B(x)$ and $\overline{\mu}_A(x) \le \overline{\mu}_B(x)$ for all x.

What we do in this paper. In this paper, we prove that in both cases – when we consider fuzzy sets and when we consider interval-valued fuzzy sets – we can indeed detect crisp sets and type-1 fuzzy sets based only on the subsethood relation $A \subseteq B$.

2 What If We Consider $[0,1]$-Based Fuzzy Sets

Our plan. To describe crisp sets in terms of the subsethood relation $A \subseteq B$, we will follow the following four steps:

- first, we will prove that the empty set $\emptyset$ can be uniquely determined based on the subsethood relation;
- second, we will show that 1-element crisp sets, i.e., sets of the type $\{x_0\}$, can be thus determined,
- third, we will prove that 1-element fuzzy sets, i.e., fuzzy sets A for which for some $x_0 \in U$, we have $\mu_A(x_0) > 0$ and $\mu_A(x) = 0$ for all $x \ne x_0$, can be determined based on the subsethood relation, and
- finally, we prove that crisp sets can be uniquely determined based on the subsethood relation.

First step: how to detect an empty set? An empty set $\emptyset$ is a fuzzy set for which $\mu_\emptyset(x) = 0$ for all $x \in U$. The detection of an empty set can be made based on the following simple result:

Proposition 1. *A fuzzy set A is an empty set if and only if $A \subseteq B$ for all fuzzy sets B.*

Proof.

1°. Let us first prove that when $A = \emptyset$, then $A \subseteq B$ for all fuzzy sets B.

Indeed, for every fuzzy set B, we have $0 \le \mu_B(x)$ for all x and thus, $\mu_\emptyset(x) = 0 \le \mu_B(x)$ for all x, i.e., we indeed have $\emptyset \subseteq B$.

2°. Let us now prove that, vice versa, if for some fuzzy set A, we have $A \subseteq B$ for every possible fuzzy set B, then $A = \emptyset$.

Indeed, in particular, the property $A \subseteq B$ is true for the case when B is the empty set. In this case, from the fact that $\mu_A(x) \le \mu_B(x) = \mu_\emptyset(x) = 0$, we conclude that $\mu_A(x) = 0$ for all x, i.e., that A is indeed the empty set.

The proposition is proven.

Second step: how to detect 1-element crisp sets based on the subsethood relation. Let us prove the following auxiliary result.

Proposition 2. *A non-empty fuzzy set A is a one-element crisp set if and only if the following two conditions are satisfied:*

- *the class $\{B : B \subseteq A\}$ is linearly ordered and*
- *for no proper superset A' of A, the class $\{B : B \subseteq A'\}$ is linearly ordered.*

Proof.

1°. Let us first prove that every 1-element crisp set, i.e., every set of the type $A = \{x_0\}$, satisfies the above two properties.

1.1°. Let us prove the first property: that the class $\{B : B \subseteq A\}$ is linearly ordered.

Indeed, for the given set A, we have $\mu_A(x_0) = 1$ and $\mu_A(x) = 0$ for all $x \ne x_0$. So, if $B \subseteq A$, i.e., if $\mu_B(x) \le \mu_A(x)$ for all x, this means that $\mu_B(x) = 0$ for all $x \ne x_0$. Thus, for such sets B, the only non-zero value of the membership function may be attained when $x = x_0$.

So, if we have two sets $B \subseteq A$ and $B' \subseteq A$, then for these two sets, $\mu_B(x) = \mu_{B'}(x) = 0$ for all $x \ne x_0$. Thus:

- if $\mu_B(x_0) \le \mu_{B'}(x_0)$, then, as one can easily check, we have $\mu_B(x) \le \mu_{B'}(x)$ for all x, i.e. we have $B \subseteq B'$, and
- if $\mu_{B'}(x_0) \le \mu_B(x_0)$, then, as one can easily check, we have $\mu_{B'}(x) \le \mu_B(x)$ for all x, we have $B' \subseteq B$.

Thus, for every two fuzzy sets B and B' from the class $\{B : B \subseteq A\}$, we have either $B \subseteq B'$ or $B' \subseteq B$. So, this class is indeed linearly ordered.

1.2°. Let us now prove that no proper superset A' of the 1-element set $A = \{x_0\}$ has the property that the class $\{B : B \subseteq A'\}$ is linearly ordered.

For the set $A = \{x_0\}$, we have $\mu_A(x_0) = 1$ and $\mu_A(x) = 0$ for all other x. If A' is a superset of A, this means that $\mu_{A'}(x) = 1$. The fact that A' is a proper superset means that $A' \ne A$, thus we have $\mu_{A'}(x') > 0$ for some $x' \ne x_0$. In this case, we can define the following fuzzy set B: $\mu_B(x') = \mu_{A'}(x')$ and $\mu_B(x) = 0$ for all $x \ne x_0$. Then, we have $B \subseteq A'$, $A \subseteq A'$, but $B \not\subseteq A$ (since $\mu_B(x') > 0$ and thus, $\mu_B(x') \not\le \mu_A(x') = 0$) and $A \not\subseteq B$ (since $1 = \mu_A(x_0) \not\le \mu_B(x_0) = 0$). Thus, the class $\{B : B \subseteq A'\}$ is indeed not linearly ordered.

2°. Let us prove that, vice versa, if a fuzzy set A has the above two properties, then it is a one-element crisp set.

2.1°. Let us first prove, by contradiction, that we can only have one element x for which $\mu_A(x) > 0$. Indeed. if $\mu_A(x_1) > 0$ and $\mu_A(x_2) > 0$ for some $x_1 \ne x_2$, then we can take the following fuzzy sets B_1 and B_2:

- $\mu_{B_1}(x_1) = \mu_A(x_1)$ and $\mu_{B_1}(x) = 0$ for all other x, and
- $\mu_{B_2}(x_2) = \mu_A(x_2)$ and $\mu_{B_2}(x) = 0$ for all other x.

Here, $B_1 \subseteq A$ and $B_2 \subseteq A$, but $B_2 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$ – which contradicts to our assumption that the class $\{B : B \subseteq A\}$ is linearly ordered.

2.2°. Due to Part 2.1, we have $\mu_A(x_0) > 0$ for at most one element x_0; for all $x \neq x_0$, we have $\mu_A(x) = 0$. Let us prove, by contradiction, that $\mu_A(x_0) = 1$, i.e., that A is indeed a one-element crisp set.

Indeed, if $\mu_A(x_0) < 1$, then we can consider the following proper superset $A' \supseteq A$: $\mu_{A'}(x_0) = (1 + \mu_A(x_0))/2 < 1$ and $\mu'_A(x) = 0$ for all other x. Similarly to Part 1.1 of this proof, we can prove that for this superset A', the class $\{B : B \subseteq A'\}$ is linearly ordered – which contradicts to our assumption that such a proper superset does not exist.

The proposition is proven.

Third step: how to detect 1-element fuzzy sets based on the subsethood relation. We say that a fuzzy set is a *1-element set* if for some $x_0 \in X$, we have $\mu_A(x_0) > 0$ and $\mu_A(x) = 0$ for all $x \neq x_0$. Let us prove the following auxiliary result.

Proposition 3. *A non-empty fuzzy set A is a one-element fuzzy set if and only if the class* $\{B : B \subseteq A\}$ *is linearly ordered.*

Proof.

1°. Arguments similar to Part 1.1 of the proof of Proposition 2 show that if A is a one-element fuzzy set, then the class $\{B : B \subseteq A\}$ is linearly ordered.

2°. Vice versa, if A is not an empty set and not a one-element fuzzy set, this means that there exist at least two values $x_1 \neq x_2$ for which $\mu_A(x_1) > 0$ and $\mu_A(x_2) > 0$. We can then take the following fuzzy sets B_1 and B_2:

- $\mu_{B_1}(x_1) = \mu_A(x_1)$ and $\mu_{B_1}(x) = 0$ for all $x \neq x_1$, and
- $\mu_{B_2}(x_2) = \mu_A(x_2)$ and $\mu_{B_2}(x) = 0$ for all $x \neq x_2$.

Then $B_1 \subseteq A$ and $B_2 \subseteq A$, but $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$. Thus, the class $\{B : B \subseteq A\}$ is not linearly ordered.

The proposition is proven.

Final result: how to detect crisp sets based on the subsethood relation. Let us prove the following auxiliary result.

Theorem 1. *A fuzzy set A is crisp if and only if every one-element fuzzy subset* $B \subseteq A$ *can be embedded in a one-element crisp subset of A.*

Comment. In other words,

$$A \text{ is crisp} \Leftrightarrow \forall A\,(B \text{ is a one-element fuzzy subset of } A \Rightarrow$$

$$\exists C\,((B \subseteq C \subseteq A)\,\&\,(C \text{ is a 1-element crisp set}))).$$

Proof.

1°. Let A be a crisp set, and let $B \subseteq A$ be a 1-element fuzzy set. By definition, this means that for some x_0, we have $\mu_B(x_0) > 0$ and $\mu_B(x) = 0$ for all other x.

Since the set A is crisp, the only possible values of $\mu_A(x_0)$ are 0 and 1. From $\mu_B(x_0) \leq \mu_A(x_0)$, we conclude that $\mu_A(x_0) > 0$ and thus, that $\mu_A(x_0) = 1$. So, $x_0 \in A$ and hence $B \subseteq \{x_0\} \subseteq A$.

2°. Vice versa, if A is not a crisp set, this means that for some element x_0, we have $0 < \mu_A(x_0) < 1$. In this case, we can take the following 1-element fuzzy set $B \subseteq A$: $\mu_B(x_0) = \mu_A(x_0)$ and $\mu_B(x) = 0$ for all $x \neq x_0$. Here, $B \subseteq A$, but the only 1-element crisp set C containing B is the set $C = \{x_0\}$, and this 1-element crisp set is *not* a subset of the original set A: $C \not\subseteq A$.

The theorem is proven.

3 What If We Consider Interval-Valued Fuzzy Sets

First step: how to detect an empty set. An empty set $\emptyset$ is an interval-valued fuzzy set for which $\mu_\emptyset(x) = [0,0]$ for all $x \in U$. The detection of an empty set can be made based on the following result:

Proposition 4. *An interval-valued fuzzy set A is an empty set if and only if $A \subseteq B$ for all interval-valued fuzzy sets B.*

Proof is similar to proof of Proposition 1.

Second step: how to detect special 1-element interval-valued fuzzy sets based on the subsethood relation. Let's introduce an auxiliary notion. We say that an interval-valued fuzzy set A is *special* if for some element x_0, we have $\mu_A(x_0) = [0,a]$ for some number $a > 0$ and $\mu_A(x) = 0$ for all $x \neq x_0$.

Proposition 5. *A non-empty interval-valued fuzzy set A is special if and only if the class $\{B : B \subseteq A\}$ is linearly ordered.*

Proof.

1°. For special sets (in the sense of the above definition), the fact that the class $\{B : B \subseteq A\}$ is linearly ordered can be proven similarly to Part 1.1 of the proof of Proposition 2.

2°. Let us now prove that, vice versa, if for some non-empty interval-valued fuzzy set A, the class $\{B : B \subseteq A\}$ is linearly ordered, then the set A is special.

2.1°. Since A is non-empty, there exists an element x_0 for which $\mu_A(x_0) \neq [0,0]$. Let us prove, by contradiction, that for every other element $x \neq x_0$, we have $\mu_A(x) = [0,0]$.

Indeed, if we had $\mu_A(x_1) \neq [0,0]$ for some $x_1 \neq x_0$, then we would be able to take the following two sets B_0 and B_1:

- $\mu_{B_0}(x_0) = \mu_A(x_0)$ and $\mu_{B_0}(x) = [0,0]$ for all $x \neq x_0$, and

- $\mu_{B_1}(x_1) = \mu_A(x_1)$ and $\mu_{B_1}(x) = [0,0]$ for all $x \neq x_1$.

In this case, $B_0 \subseteq A$ and $B_1 \subseteq A$, but $B_0 \not\subseteq B_1$ and $B_1 \not\subseteq B_0$. This contradicts our assumption that the class $\{B : B \subseteq A\}$ is linearly ordered.

2.2°. To complete the proof of the proposition, we need to prove that the value $\mu_A(x_0) = [\underline{\mu}_A(x_0), \overline{\mu}_A(x_0)]$ has the form $[0,a]$ for some $a > 0$, i.e., that $\underline{\mu}_A(x_0) = 0$.

We will prove it by contradiction. Suppose that, vice versa, $\underline{\mu}_A(x_0) > 0$. In this case, we can take the following sets B_1 and B_2:

- $\mu_{B_1}(x_0) = [0.5 \cdot \underline{\mu}_A(x_0), 0.5 \cdot \underline{\mu}_A(x_0)]$ and $\mu_{B_1}(x) = 0$ for all $x \neq x_0$, and
- $\mu_{B_2}(x_0) = [0, \underline{\mu}_A(x_0)]$ and $\mu_{B_2}(x) = 0$ for all $x \neq x_0$.

Then, $B_1 \subseteq A$ and $B_2 \subseteq A$, but $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$. This contradicts our assumption that the class $\{B : B \subseteq A\}$ is linearly ordered.

The proposition is proven.

Third step: how to detect 1-element type-1 fuzzy sets based on the subsethood relation. We say that an interval-valued fuzzy set is a *1-element type-1* fuzzy set if there exists an element x_0 for which $\mu_A(x_0) = [a,a]$ for some $a > 0$ and $\mu_A(x) = [0,0]$ for all $x \neq x_0$.

Proposition 6. *A non-empty interval-valued fuzzy set A is a 1-element type-1 set if and only if it is satisfies the following three properties:*

- *the set A is not special (in the sense of the above definition),*
- *there exists a special set $B \subseteq A$ for which the class $\{C : B \subseteq C \subseteq A\}$ is linearly ordered, and*
- *for no proper superset A' of A, the class $\{C : B \subseteq C \subseteq A\}$ is linearly ordered.*

Proof is similar to the proof of Proposition 2.

Final result. Since we have subsethood, we also have union: the union of A_α is the $\subseteq$-smallest set that contains all $A - \alpha$. We can thus define type-1 fuzzy sets as unions of 1-element type-1 fuzzy sets. Once we can detect type-1 fuzzy sets, we can use techniques from the previous section to detect crisp sets. Thus, *we can indeed detect type-1 fuzzy sets and crisp sets based only on subsethood relation between interval-valued fuzzy sets.*

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