Contradictions Do Not Necessarily Make a Theory Inconsistent

Olga Kosheleva and Vladik Kreinovich
University of Texas at El Paso
500 W. University
El Paso, Texas 79968, USA
olgak@utep.edu, vladik@utep.edu

Abstract

Some religious scholars claim that while the corresponding holy texts may be contradictory, they lead to a consistent set of ethical and behavioral recommendations. Is this logically possible? In this paper, somewhat surprisingly, we kind of show that this is indeed possible: namely, we show that if we add, to statements about objects from a certain class, consequences of both contradictory abstract statements, we still retain a consistent theory. A more mundane example of the same phenomenon comes from mathematics: if we have a set-theoretical statement S which is independent from ZF and which is not equivalent to any arithmetic statement, then we can add both arithmetic statements derived from S and arithmetic statements derived from $\neg S$ and still keep the resulting class of arithmetic statements consistent.

1 Can Contradictions Lead to a Consistent Theory?

There are seeming logical contradictions in holy books. From the purely logical viewpoint, holy books often contain inconsistent statements. For example, the Bible has two different stories of creation:

- in one, Adam was created first and Eve made out of his rib later on – since he felt lonely in the Paradise, while
- in the second one, both first humans were created at the same time.

How religions treat such contradictions. There are two main approaches to such seeming contradictions. The first approach is to try to re-interpret the text so that the contradictions disappear.

Interestingly, there is also a second approach (see, e.g., [7]), that yes, from our viewpoint, this may be perceived as a contradiction, but both contradictory statements have important consequences about human behavior – and when we restrict ourselves to behavioral, ethical consequences of both contradictory statements, then there is no inconsistency anymore.

For example, we can interpret the second creation story as implying that, since men and women were created at the same time, they are equal and should be treated as equal. On the other hand, we can view the first story as emphasizing the need for human companionship, companionship that brings benefits even if we sacrifice something for it (like Adam sacrificed a rib).

A natural question: does this second approach make logical sense? From the precise logical viewpoint, is it possible to include both contradictory statements and keep the resulting theory consistent – as long as we talk about consequences from a certain class (in this case, behavioral consequences)? If this is logically possible, this would provide a logical justification for the second approach:

- yes, on the abstract and/or historical level, there is a contradiction,
- but if we only consider behavioral consequences, conclusions are perfectly consistent.

What we show in this paper. In this paper, we show, somewhat surprisingly, that the above "second approach" is indeed logically possible: we *can* have a consistent sub-theory that uses both contradictory statements from the general theory.

Somewhat surprisingly, a similar situation occurs in mathematics. At first glance, in mathematics, when we consider consistent theories, there is no room for contradictions. But there is.

While mathematicians analyze and explore many abstract objects, a special interest is still in analyzing simple objects such as finite graphs, integers, real numbers, etc. This is why most abstract objects were invented: they have been useful in proving results about the traditional objects such as graphs, integers, and real numbers. For example (see, e.g., [1]):

- Georg Cantor invented set theory largely to solve open problems about Fourier series,
- many abstract algebraic objects such as groups came from attempts to find general formulas for solving polynomial equations, etc.

The need to go beyond the existing theory of real numbers, integers, etc., comes directly from the well-known Gödel's theorem, according to which, no matter how many axioms we assume, there is always a statement about natural numbers that cannot be neither proven nor disproven from these axioms. In particular, there are arithmetic statements that cannot be proven or disproven based on the current axiomatization ZF of set theory.

A natural way to deal with these statements is to go beyond ZF. Indeed, many additional set-theoretical axioms have been proposed that are independent from ZF and that enable us to go beyond ZF, e.g.:

- statements about the existence of so-called large cardinals (larger than anything that can be constructed in ZF) or
- opposite statements limiting set theory to what can be, in some reasonable sense, constructed in ZF;

(see, e.g., [4] for technical details).

Some such statements are helpful in proving arithmetic results. Interestingly, sometimes, an additional statement S helps to prove some arithmetic results, while its negation $\neg S$ helps prove some other arithmetic results.

Traditional reasoning is that since S and $\neg S$ are inconsistent, we have to choose one or another. But would not it be nice to be able to keep consequences of *both* S and $\neg S$?

What we show in this paper. In this paper, we show that this is indeed possible: we can have a consistent arithmetic theory even if we add consequences of both statements S and $\neg S$.

2 Definitions and the Main Result

Need for precise definitions. To formulate and prove our main result, we need to recall relevant definitions.

What is a statement. As usual, by a *statement*, we will understand a first order logical statement in an appropriate language, i.e., anything that can be obtained from some basic predicates $P_i(a, \ldots, b)$ by using logical connectives "and" (&), "or" ($\vee$), "not" ($\neg$), "implies" ($\rightarrow$) and quantifiers $\forall x$ and $\exists x$.

For example:

- in the theory of natural numbers, the basic predicates may be $a = b$, $a = b + 1$, $a = b + c$, $a = b \cdot c$, $a < b$, etc.;
- in set theory, the basic predicates as $a = b$, $a \in b$, and, e.g., $a \subseteq b$, etc.

The set of all possible statements will denoted by L.

What is a theory. By a *theory* T, we mean a set of statements from a given language L which can be derived from a certain list of statements called *axioms*. In other words, a theory is a set of statement closed under usual deduction rules – such as

if A is true and $A \rightarrow B$ is true then B is also true;

i.e., that:

$$\text{if } A \in T \text{ and } (A \rightarrow B) \in T, \text{ then } B \in T.$$

We will also consider a sub-theory of the original theory. Let T be a theory. Within this theory, we can only consider objects that satisfy a certain property $P(x)$, and only consider statements about such objects. We will call such objects *P-objects.*

For example:

- within a general set theory T, we can only consider objects which are natural numbers;
- within a general theory about the world, we only consider statements about possible human actions.

Statements about P-objects will be called *P-statements.* Let us denote the class of all P-statements from the original theory T by T_P.

Use of contradictory statements. Let us assume that in the language L, we have a statement S, which is *independent* on T – i.e., neither S nor its negation belongs to the theory T.

Let us also assume that the statement S is not equivalent to any statement about the P-objects. Our claim is that in this case, we can add, to T_P, all consequences of *both* S and $\neg S$ related to P-objects, and still retain a consistent theory of P-objects.

Definition. *Let L be a language and let P be a predicate in L.*

- *Objects that satisfy the property P are called P-*objects*.*
- *A statement is called a P-*statement *if it is about P-objects.*

Let T be a theory in the language L. For every property P, by T_P, we will then denote the set of all the P-statements from T.

Proposition. *Let T be a theory, let P be a property and let S be a statement from L such that:*

- *S is independent from T and*
- *S is not equivalent to any P-statement.*

Then, if we add, to T_P, all P-statements that follow from S and all P-statements that follow from $\neg S$, we will still get a consistent theory of P-objects.

Proof. In this proof, we will use the basic facts and ideas from mathematical logic; see, e.g., [2, 5].

We will prove the desired statement by contradiction. Let us assume that after adding consequences of S and consequences of $\neg S$, we get a contradiction. In other words, we adding P-statements $A_1, \ldots, A_n$ which follow from S and P-statements $B_1, \ldots, B_m$ which follows from $\neg S$, we get a contradiction $\top$.

So, for the P-statements $A \stackrel{\text{def}}{=} A_1 \,\&\, \ldots \,\&\, A_n$ and $B \stackrel{\text{def}}{=} B_1 \,\&\, \ldots \,\&\, B_m$, we get

$$S \to A, \tag{1}$$

$$\neg S \to B, \tag{2}$$

and

$$A \,\&\, B \to \top. \tag{3}$$

From the implication (3), it follows that

$$A \to \neg B. \tag{4}$$

From the implication (2), it follows that

$$\neg B \to S. \tag{5}$$

Combining implications (1), (4), and (5), we get

$$S \to A \to \neg B \to S, \tag{6}$$

hence

$$S \to A \text{ and } A \to S.$$

Therefore, the statement S is equivalent to a P-statement A, which contradicts to our assumption that S is not equivalent to any P-statement.

This contradiction proves the proposition.

Comment. A similar proof appeared previously – albeit in a different context – in [3, 6].

Acknowledgments

This work was supported in part by the National Science Foundation grants HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and DUE-0926721, and by an award "UTEP and Prudential Actuarial Science Academy and Pipeline Initiative" from Prudential Foundation.

References

[1] C. B. Boyer and U. C. Merzbach, *A History of Mathematics*, Wiley, Hoboken, New Jersey, 2011

[2] H. B. Enderton, *A Mathematical Introduction to Logic*, Academic Press, New York, 2001.

[3] V. Kreinovich. *Should We Add a Formula Or Its Negation to an Incomplete Theory? Or Both?*, University of Texas at El Paso, Computer Science Department, Technical Report No. UTEP-CS-90-13, July 1990.

[4] K. Kunen, *Set Theory*, College Publications, London, UK, 2011.

[5] J. R. Schoenfeld, *Mathematical Logic*, Addison-Wesley, 2001.

[6] H. T. Nguyen and V. Kreinovich, "Classical-Logic analogue of a fuzzy 'paradox' ", *Proceedings of the 1996 IEEE International Conference on Fuzzy Systems*, New Orleans, September 8–11, 1996, Vol. 1, pp. 428–431.

[7] J. B. Soloveitchik, *The Lonely Man of Faith*, Jason Aronson Publ., Northvale, New Jersey, 1997.