The Onsager Conjecture: A Pedagogical Explanation

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Abstract

In 1949, a Nobelist Last Onsager considered liquid flows with velocities changing as r^α for spatial points at distance r, and conjectured that the threshold value $\alpha = 1/3$ separates the two possible regimes: for $\alpha > 1/3$ energy is always preserved, while for $\alpha < 1/3$ energy is possibly not preserved. In this paper, we provide a simple pedagogical explanation for this conjecture.

1 Formulation of the Problem

The equations that describe the velocity filed $v(t,x)$ of an incompressible non-viscous liquid go back to Euler – and are known as *Euler equations*. In 1949, a Nobelist Lars Onsager considered solutions $v(t,x)$ to Euler's equation for which, for some constant C, we have $|v(t,x) - v(t,x')| \le C \cdot r^\alpha$, where r denotes the distance between the points x and x' [2]. He conjectured that:

- when $\alpha > 1/3$, then all the corresponding solutions $v(t,x)$ preserve energy, while
- for $\alpha < 1/3$, there exist solutions that do not preserve energy.

This conjecture remains one of the central open problems in the foundations of hydrodynamics; see, e.g., [1] and references therein.

How can we explain this technical conjecture in simple physical terms? The main objective of this paper is to provide such an explanation.

2 Our Explanation

Starry sky and turbulent sea: two basic examples of dynamical situations. Let us forget for a while about the modern physics with its measuring instruments, let us go back to basic phenomena which are observable with a naked eye. Most of the things that we see in the words change very slowly: seasons change year after year, whether changes from months to months, flowers blossom, rivers have annual floods and droughts, etc. But there are two major things that change reasonably fast, so that we can see the changes in terms of hours or even seconds:

- the stars (and planets) in the sky and
- the waves on the sea surface.

What is the main difference between the starry sky and the turbulent sea? In the sky, at any given moment of time, the picture is very simple: just a few bright dots against the black sky. However, with respect to time, the changes are very complex: it took Newton's theory of gravitation to finally explain this complex motion.

On the other hand, for the waves on the seashore, the situation is exactly opposite. If we fix a location and trace how the waves change with time, we get a simple periodic dependence. In contrast, the *spatial* shape of the wave patterns can be very complex.

In other words:

- for the starry skies, the spatial picture is simple, but the temporal dependence is complex, while
- for the waves in the sea, the temporal picture is simple, but the spatial picture is complex.

So maybe if we swap time and space, we get the relation between the starry skies and the waves in the sea? How can we relate the starry skies and the waves in the sea? It makes sense to relate the complex with the complex and the simple with the simple. In view of the above, this means:

- that we relate the temporal picture of the starry skies with the spatial picture of the waves, and
- that relate the spatial picture of the starry skies with the temporal picture of the waves.

In other words, to properly relate the two phenomena, we need to "swap" space and time.

Let us use this idea to translate known laws of starry skies into conjectures about the waves. Thanks to Newton's laws, we know how to describe the motion of celestial bodies. Our knowledge of the waves is much less complete. Let us therefore use the time-space swap idea to translate knowledge about celestial bodies into knowledge about the waves.

For that, let us briefly recall how we can describe the motion of celestial bodies.

Newton's theory of gravitation: a brief reminder. According to Newton's theory, the acceleration a caused by a fixed body is inverse proportional to the square of the distance or this body: $a = \frac{C}{r^2}$. What is acceleration? It is a description of how velocity v changes with time: $a \approx \frac{\Delta v}{\Delta t}$. So, the above formula, in effect, means that

$$\frac{\Delta v}{\Delta t} = \frac{C}{r^2},$$

i.e., that

$$\Delta v = C \cdot \frac{\Delta t}{r^2}. \tag{1}$$

What is the scale-invariance of this formula? The numerical values of all physical quantities depend on what measuring unit we use. If we use centimeters instead of meters, all the numerical values of distance are increased by a factor of 100. Similarly, if we replace minutes by seconds, then the numerical values of all time intervals increase by a factor of 60.

In general, if we replace the unit of length with the λ times smaller one, then all the numerical values of the distance change from r to $\lambda \cdot r$. Similarly, if we replace the unit of time with the μ times smaller one, then all the numerical values of the time interval change from Δt to $\mu \cdot \Delta t$. As a result, e.g., numerical values of the velocity $v = \frac{\Delta x}{\Delta t}$ changes from v to $\frac{\lambda}{\mu} \cdot v$.

Physical laws do not depend on the choice of the units. This does not mean, of course, that if we simply replace r with $\lambda \cdot r$, all physical formulas remain the same: one can easily see that, e.g., the above formula (1) will not remain the same under this change. What is true is that if we change the unit of distance *and* appropriately change the unit of time, then the formula (1) indeed remains invariant. Namely, in the formula (1):

- the left-hand side is multiplied by the factor $\frac{\lambda}{\mu^2}$, while
- the right-hand side changes by the factor $\frac{1}{\lambda^2}$.

For the equality to remain valid, these two factors must coincide, i.e., we must have $\frac{\lambda}{\mu^2} = \frac{1}{\lambda^2}$ and thus, $\lambda^3 = \mu^2$ and $\mu = \lambda^{3/2}$. This means that time intervals change in exactly the same way as distance to the power of 3/2: $t \sim r^{3/2}$.

Consequence of Newton's laws: phenomenon of the critical velocity. It is known that for an object to follow a circular orbit at a distance r from the central body of mass M, we need the velocity v for which $v^2 = \text{const} \cdot \frac{M}{r}$.

- If the velocity is higher, the object will fly away from this orbit.
- If the velocity is lower, the object will fall down on the central body.

The circular-motion velocity is thus a threshold velocity separating two different regimes.

The formula for this threshold velocity is invariant with respect to the same re-scaling as the Newton's formula for the gravitational force: $\mu = \lambda^{3/2}$.

What can we then conclude about waves? As we have mentioned, a reasonable idea is to take the description of celestial bodies and swap time t and spatial distance r. If we do this swap, then instead of $t \sim r^{3/2}$ we get $x \sim t^{3/2}$. In other words, the physical laws that describe the waves should remain invariant if we apply the re-scalings with $\lambda = \mu^{3/2}$.

In the description of celestial bodies, the most complex phenomenon is how velocities changes with time, so we were interested in the change

$$\Delta v = v(t + \Delta t, x) - v(t, x)$$

with time. For the waves, the most complex phenomenon is how the velocity changes in space, we are interested in how the velocity changes from one spatia location to another one. In other words, we are interested in the spatial differences $\Delta v = v(t, x + \Delta x) - v(t, x)$.

So, we need to find the dependence of Δv on the distance r between the points $x + \Delta x$ and x: $\Delta v = f(r)$. When is this dependence invariant under the above re-scaling? Under this re-scaling, the value Δv is multiplied by

$$\frac{\lambda}{\mu} = \frac{\mu^{3/2}}{\mu} = \mu^{1/2}.$$

Thus, the right-hand side must multiply by the same factor, i.e., we must have

$$f(\lambda \cdot r) = f(\mu^{3/2} \cdot r) = \mu^{1/2} \cdot f(r).$$

For $r = 1$ and $x = \mu^{3/2}$ – so that $\mu = x^{2/3}$ – we get

$$f(x) = \mu^{1/2} \cdot f(1) = (x^{2/3})^{1/2} \cdot f(1) = x^{1/3} \cdot f(1).$$

Thus, we conclude that $f(r) = C \cdot r^{1/3}$, where we denoted $C \stackrel{\text{def}}{=} f(1)$, and we get

$$\Delta v \sim r^{1/3}.$$

This is exactly the threshold dependence separating the two regimes according to the Onsager's conjecture.

Thus, we have indeed come up with a reasonable explanation for this conjecture.

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References

[1] S. Klainerman, "Nash's contribution to analysis in three papers", *Bulletin of the American Methematical Society*, 2017, Vol. 54, No. 2, pp. 283–305.

[2] L. Onsager, "Statistical hydrodynamics", *Nuovo Cimento, Ser. IX*, 1949, Vol. 6, Supplement No. 2, pp. 279–287.