Do It Today Or Do It Tomorrow: Empirical Non-Exponential Discounting Explained by Symmetry Ideas

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Abstract. At first glance, it seems to make sense to conclude that when a 1 dollar reward tomorrow is equivalent to a $D < 1$ dollar reward today, the day-after-tomorrow's 1 dollar reward would be equivalent to $D \cdot D = D^2$ dollars today, and, in general, a reward after time t is equivalent to $D(t) = D^t$ dollars today. This *exponential discounting* function $D(t)$ was indeed proposed by the economists, but it does not reflect the actual human behavior. Indeed, according to this formula, the effect of distant future events is negligible, and thus, it would be reasonable for a person to take on huge loans or get engaged in unhealthy behavior even when the long-term consequences will be disastrous. In real life, few people behave like that, since the actual empirical discounting function is different: it is hyperbolic $D(t) = 1/(1 + k \cdot t)$. In this paper, we use symmetry ideas to explain this empirical phenomenon.

1 Discounting: Theoretical Foundations, Empirical Data, and Related Challenge

What is discounting. Future awards are less valuable than the same size awards given now. This phenomenon is known as *discounting*; see, e.g., [1, 3, 5–9, 11] for details.

Procrastination is an inevitable consequence of discounting. Suppose that we have a task which is due by a certain deadline. This can be a task of submitting a grant proposal, or of submitting a paper to a conference.

In this case, the reward is the same no matter when we finish this task – as long as we finish it before the deadline. Similarly, the overall negative effect caused by the need to do some boring stuff is the same no matter when we do it. But, due to discounting, if we perform this task later, today's negative effect is smaller than if we perform this task today. The further in the future is this negative effect, the smaller is its influence on our today's happiness. Thus, a natural way to maximize today's happiness is to postpone this task as much as possible – which is exactly what people do; see, e.g., [2, 7].

A simple theoretical model of discounting. How can we describe discounting in numerical terms? At first glance, providing numerical description for discounting is a straightforward idea.

Indeed, let us assume that 1 dollar tomorrow is equivalent to $D < 1$ dollars today. This is true for every day: 1 dollar at the day $t + 1$ is equivalent to D dollars in day t.

This means, in particular, that 1 dollar in day t_0+2 is equivalent to D dollars at time t_0+1. Since 1 dollar on day t_0+1 is equivalent to D dollars at the initial moment of time t_0, D dollars on day $t_0 + 1$ are equivalent to $D \cdot D$ dollars on day t_0. Thus, we can conclude that 1 dollar at day $t_0 + 2$ is equivalent to D^2 dollars at moment t_0.

Similarly, 1 dollar at moment $t_0 + 3$ is equivalent to D dollars at moment $t_0 + 2$ and thus, to $D \cdot D^2 = D^3$ dollars at moment t_0. In general, by induction over t, we can show that 1 dollar at moment $t_0 + t$ is equivalent to $D(t) \stackrel{\text{def}}{=} D^t$ dollars at the current moment t_0.

We can rewrite the above expression $D(t) = D^t$ as

$$D(t) = \exp(-a \cdot t), \tag{1}$$

where $a \stackrel{\text{def}}{=} -\ln(D)$. Because of this form, this discounting is known as *exponential.*

Practical problem with exponential discounting. At first glance, exponential discounting is a very reasonable idea. However, it has a problem: exponential functions decrease very fast, and for large t, the value $\exp(-a \cdot t)$ becomes indistinguishable from 0.

In practical terms, this means that a person looks for an immediate reward even if there is a significant negative downside in the distant future.

Such behavior indeed happens: a young man takes many loans without taking into account that in the future, he will have to pay; a young person ruins his health by using drugs, not taking into account that in the future, this may lead to a premature death. A person commits a crime without taking into consideration that eventually, he will be caught and punished.

Such behavior does happen, but such behavior is abnormal. Most people do not take an unrealistic amount of loans, most people do not ruin their health during their youth, most people do not commit crimes. This means that for most people, discounting decreases much slower than the exponential function.

So how to describe discounting: empirical data. Empirical data shows that discounting indeed decreases much slower than predicted by the exponential function: namely, 1 dollar at moment $t_0 + t$ is equivalent to

$$D(t) = \frac{1}{1 + k \cdot t} \tag{2}$$

dollars at moment t_0. This formula is known as *hyperbolic discounting*; see, e.g., [1, 3, 5–9, 11].

Problem: how can we explain the empirical data. In principle, there exist many functions that decrease slower than the exponential function $\exp(-a \cdot t)$. So why, out of all these functions, we observe the hyperbolic one?

What we do in this paper. In this paper, we use symmetries to provide a theoretical explanation for the empirical discounting formula. To be more precise, our theoretical explanation leads to a family of functions of which hyperbolic discounting is one of the possibilities.

2 Analysis of the Problem

The idea of a re-scaling. Let $D(t)$ denote the discounting of a reward which is t moments into the future, i.e., the amount of money such that getting $D(t)$ dollars now is equivalent to getting 1 dollar after time t.

By definition, $D(0)$ means getting 1 dollar with no delay, so $D(0) = 1$. It is also reasonable to require that as the time period time t increases, the value of the reward goes to 0, so that $\lim\limits_{t \to +\infty} D(t) = 0$.

It is also reasonable to require that a small change in t should lead to small changes in $D(t)$, i.e., that the function $D(t)$ be differentiable (smooth).

The further into the future we get the reward, the less valuable this reward is now, so the function $D(t)$ is decreasing as the time t increases. Thus, if we further delay all the rewards by some time s, then each value $D(t)$ will be replaced by a smaller value $D(t+s)$. We can describe this replacement as $D(t+s) = F_s(D(t))$, where the function $F_s(x)$ re-scales the original discount value $D(t)$ into the new discount value $D(t+s)$.

For the exponential discounting (1), the re-scaling $F_s(x)$ is linear: $D(t+s) = C \cdot D(t)$, where $C \stackrel{\text{def}}{=} \exp(-a \cdot s)$, so we have $F_s(x) = C \cdot x$. For the hyperbolic discounting (2), the corresponding re-scaling $F_s(x)$ is not linear.

Which re-scaling should we select?

Which re-scalings are reasonable: formulating this question in precise mathematical terms. We want to select some *reasonable* re-scalings. What does "reasonable" mean? Of course, linear re-scalings should be reasonable.

Also, intuitively, if a re-scaling is reasonable, then its inverse should also be reasonable. Similarly, if two re-scalings are reasonable, then applying them one after another should also lead to a reasonable re-scaling. In other words, a composition of two re-scalings should also be reasonable. In mathematical terms, we can conclude that the class of all reasonable re-scalings should be closed under inverse transformation and composition of two mappings. This means that with respect to the composition operation, such re-scalings must form a *group*.

We want to be able to determine the transformation from this group based on finitely many experiments. In each experiment, we gain a finite number of values, so after a finite number of experiments, we can only determine a finite number of parameters. Thus, we should be able to select an element of the desired transformation group based on the values of finitely many parameters.

In mathematical terms, this means that the corresponding transformation group should be *finite-dimensional*.

Summarizing: we want all the transformations $F_s(x)$ to belong to a finite-dimensional transformation group of functions of one variable that contains all linear transformations.

Which re-scalings are reasonable: answer to the question. It is known (see, e.g., [4, 10, 12]) that the only finite-dimensional transformation groups of functions of one variable that contain all linear transformations are the group of all linear transformations and the group of all fractional-linear transformations

$$\frac{a + b \cdot x}{1 + c \cdot x}.$$

Thus, our informal requirement that each re-scaling is reasonable implies that each re-scaling should be fractionally linear:

$$F_s(x) = \frac{a(s) + b(s) \cdot x}{1 + c(s) \cdot x}.$$

So, we arrive at the following requirement.

3 Definition and the Main Result

Definition 1. *We say that a smooth decreasing function $D(t)$ for which $D(0) = 1$ and $\lim\limits_{t \to \infty} D(t) = 0$ is a* reasonable discounting function *if for every s, there exist values $a(s)$, $b(s)$, and $c(s)$ for which*

$$D(t + s) = \frac{a(s) + b(s) \cdot D(t)}{1 + c(s) \cdot D(t)}. \tag{3}$$

Proposition 1. *A function $D(t)$ is a reasonable discounting function if and only if it has one of the following forms: $D(t) = \exp(-a \cdot t)$, $D(t) = \dfrac{1}{1 + k \cdot t}$, $D(t) = \dfrac{1 + a}{1 + a \cdot \exp(k \cdot t)}$, or $D(t) = \dfrac{a}{(a + 1) \cdot \exp(k \cdot t) - 1}$, for some $a > 0$ and $k > 0$.*

Comment. The first discounting function corresponds to exponential discounting, the second to the hyperbolic discounting, the other two functions correspond to the more general case.

Both exponential and hyperbolic discounting can be viewed as the limit case of the more general formulas. Indeed, in the limit $a \to \infty$, both general expressions tend to the formula $D(t) = \exp(-k \cdot t)$ corresponding to the exponential discounting.

On the other hand, if we tend k to 0, we get $\exp(k \cdot t) \approx 1 + k \cdot t$, so for $a(k) = \alpha \cdot k$, the second general formula takes the form

$$D(t) = \frac{\alpha \cdot k}{(1 + \alpha \cdot k) \cdot (1 + k \cdot t) - 1}.$$

The denominator of this expression has the form

$$1 + \alpha \cdot k + k \cdot t + \alpha \cdot k^2 \cdot t - 1 = \alpha \cdot k + (k + \alpha \cdot k^2) \cdot t,$$

so

$$D(t) = \frac{\alpha \cdot k}{\alpha \cdot k + (k + \alpha \cdot k^2) \cdot t}.$$

Dividing both numerator and denominator of this formula by $\alpha \cdot k$, we get the hyperbolic discounting $D(t) = \frac{1}{1 + k' \cdot t}$, with $k' = k + \frac{1}{\alpha}$, i.e., in the limit $k \to 0$, with $k' = \frac{1}{\alpha}$.

Proof.

1°. Let us first show that each of the four functions $D(t)$ listed in the formulation of the Proposition is a reasonable discounting function in the sense of Definition 1.

It is easy to see that all four functions are smooth and decreasing, and that for all of them, we have $D(0) = 1$ and $\lim\limits_{t \to \infty} D(t) = 0$. Let us show, one by one, that each of these four functions satisfies the property (3) for appropriate auxiliary functions $a(s)$, $b(s)$, $c(s)$, and $d(s)$.

1.1°. For $D(t) = \exp(-a \cdot t)$, we have $\exp(a - a \cdot (t + s)) = \exp(-a \cdot s) \cdot \exp(-a \cdot t)$, i.e., $D(t + s) = \exp(-a \cdot s) \cdot D(t)$. Thus, the condition (3) is satisfied for $a(s) = 0$, $b(s) = \exp(-a \cdot s)$, and $c(s) = 0$.

1.2°. For the function $D(t) = \frac{1}{1 + k \cdot t}$, we have $1 + k \cdot t = \frac{1}{D(t)}$ hence

$$1 + k \cdot (t + s) = (1 + k \cdot t) + k \cdot s = \frac{1}{D(t)} + k \cdot s$$

and thus,

$$D(t + s) = \frac{1}{1 + k \cdot (t + s)} = \frac{1}{\frac{1}{D(t)} + k \cdot s}.$$

Multiplying both numerator and denominator of the right-hand side by $D(t)$, we conclude that

$$D(t + s) = \frac{D(t)}{1 + k \cdot s \cdot D(t)}.$$

Thus, the condition (3) is satisfied for $a(s) = 0$, $b(s) = 1$, and $c(s) = k \cdot s$.

1.3°. For the function $D(t)=\dfrac{1+a}{1+a\cdot\exp(k\cdot t)}$, we have $1+a\cdot\exp(k\cdot t)=\dfrac{1+a}{D(t)}$. Thus,

$$a\cdot\exp(k\cdot t)=\frac{1+a}{D(t)}-1=\frac{1+a-D(t)}{D(t)},$$

hence

$$a\cdot\exp(k\cdot(t+s))=\exp(k\cdot s)\cdot(a\cdot\exp(k\cdot t))=\exp(k\cdot s)\cdot\frac{1+a-D(t)}{D(t)}=$$

$$\frac{\exp(k\cdot s)\cdot(1+a)-\exp(k\cdot s)\cdot D(t)}{D(t)}.$$

Thus,

$$1+a\cdot\exp(k\cdot(t+s))=1+\frac{\exp(k\cdot s)\cdot(1+a)-\exp(k\cdot s)\cdot D(t)}{D(t)}=$$

$$\frac{\exp(k\cdot s)\cdot(1+a)-(\exp(k\cdot s)-1)\cdot D(t)}{D(t)}.$$

Therefore,

$$D(t+s)=\frac{1+a}{1+a\cdot\exp(k\cdot(t+s))}=$$

$$\frac{(1+a)\cdot D(t)}{\exp(k\cdot s)\cdot(1+a)-(\exp(k\cdot s)-1)\cdot D(t)}.$$

If we divide both the numerator and the denominator of this fraction by

$$(1+a)\cdot\exp(k\cdot s),$$

we conclude that

$$D(t+s)=\frac{\exp(-k\cdot s)\cdot D(t)}{1-\dfrac{1-\exp(-k\cdot s)}{1+a}\cdot D(t)}.$$

This is a formula of type (3), with $a(s)=0$, $b(s)=\exp(-k\cdot s)$, and

$$c(s)=-\frac{1-\exp(-k\cdot s)}{1+a}.$$

1.4°. Finally, for the function $D(t)=\dfrac{a}{(a+1)\cdot\exp(k\cdot t)-1}$,, we have

$$(a+1)\cdot\exp(k\cdot t)-1=\frac{1}{D(t)},$$

hence

$$(a+1)\cdot\exp(k\cdot t)=\frac{1}{D(t)}+1=\frac{a+D(t)}{D(t)}.$$

Thus,

$$(a+1)\cdot\exp(k\cdot(t+s)) = \exp(k\cdot s)\cdot(a\cdot\exp(k\cdot t)) = \exp(k\cdot s)\cdot\frac{a+D(t)}{D(t)} =$$

$$\frac{\exp(k\cdot s)\cdot a+\exp(k\cdot s)\cdot D(t)}{D(t)}.$$

So, we have

$$(a+1)\cdot\exp(k\cdot(t+s))+1 = \frac{\exp(k\cdot s)\cdot a+\exp(k\cdot s)\cdot D(t)}{D(t)}+1 =$$

$$\frac{\exp(k\cdot s)\cdot a+(\exp(k\cdot s)+1)\cdot D(t)}{D(t)}.$$

Thus,

$$D(t+s) = \frac{a}{(a+1)\cdot\exp(k\cdot(t+s))+1} =$$

$$\frac{a\cdot D(t)}{\exp(k\cdot s)\cdot a+(\exp(k\cdot s)+1)\cdot D(t)}.$$

If we divide both the numerator and the denominator of this fraction by

$$\exp(k\cdot s)\cdot a,$$

we conclude that

$$D(t+s) = \frac{\exp(-k\cdot s)\cdot D(t)}{1+\dfrac{1+\exp(-k\cdot s)}{a}\cdot D(t)}.$$

This is a formula of type (3), with $a(s)=0$, $b(s)=\exp(-k\cdot s)$, and

$$c(s) = \frac{1+\exp(-k\cdot s)}{a}.$$

2°. So, for all four cases, we have proved that the corresponding functions $D(t)$ are reasonable discounting functions. Let us now show that, vice versa, if $D(t)$ is a reasonable discounting function in the sense of Definition 1, then it has one of the four forms listed in the formulation of Proposition 1.

2.1°. Let us assume that $D(t)$ is a reasonable discounting function. Thus, by definition of a reasonable discounting function, there exists functions $a(s)$, $b(s)$, and $c(s)$ for which, for all t and s, we have the property (3). Let us first prove that in this case, we have $a(s)=0$.

Indeed, in the limit when $t\to\infty$, we have $D(t)\to 0$ and $D(t+s)\to 0$. Tending both sides of the equality (3) to the limit, we conclude that $a(s)=0$. Thus, the formula (3) can be reformulated in an equivalent simplified form:

$$D(t+s) = \frac{b(s)\cdot D(t)}{1+c(s)\cdot D(t)}. \tag{4}$$

2.2°. Let us now prove that the functions $b(s)$ and $c(s)$ are differentiable.

By definition of a reasonable discounting function, the function $D(t)$ is differentiable. Now, if we multiply both sides of the formula (4) by the denominator of the right-hand side, we get:

$$D(t+s) + c(s) \cdot D(t) \cdot D(t+s) = b(s) \cdot D(t).$$

If we now move all the terms containing $b(s)$ and $D(s)$ to the left-hand side and all the other terms to the right-hand side, we conclude that

$$b(s) \cdot (-D(t)) + c(s) \cdot (D(t) \cdot D(t+s)) = -D(t+s).$$

Thus, for each s, if we take two different values $t = t_1$ and $t = t_2$, we will get a system of two linear equations from which we can determine the two unknowns $b(s)$ and $c(s)$:

$$b(s) \cdot (-D(t_1)) + c(s) \cdot (D(t_1) \cdot D(t_1+s)) = -D(t_1+s);$$

$$b(s) \cdot (-D(t_2)) + c(s) \cdot (D(t_2) \cdot D(t_2+s)) = -D(t_2+s).$$

The solution of a system of linear equation can be explicitly described by the Cramer's rule. According to this rule, we have a differentiable formula that describes the solution to a system in terms of the coefficients at the unknowns and of the right-hand sides. In our case, both coefficients and right-hand sides are differentiable functions of s – since the discounting function $D(t)$ is differentiable. Thus, we conclude that the functions $b(s)$ and $c(s)$ are also differentiable.

2.3°. Let us now use the fact all the functions $D(t)$, $b(s)$, and $c(s)$ are differentiable to transform a difficult-to-solve functional equation (4) into a easier-to-solve differential equation. For this purpose, let us differentiate both sides of the formula (4) with respect to s and take $s = 0$.

After differentiation, we get the following differential equation:

$$D'(t+s) = \frac{b'(s) \cdot D(t)}{1 + c(s) \cdot D(t)} - \frac{b(s) \cdot D(t) \cdot c'(s) \cdot D(t)}{(1 + c(s) \cdot D(t))^2},$$

where b', c', and D' denote the derivatives of the corresponding functions. Let us now take $s = 0$. In this case, $D(t+s) = D(t)$, so we have $b(0) = 1$ and $c(0) = 1$. Thus, we get:

$$D'(t) = b'(0) \cdot D(t) - c'(0) \cdot (D(t))^2,$$

i.e.,

$$\frac{dD}{dt} = B \cdot D - C \cdot D^2, \tag{5}$$

where we denoted $B \stackrel{\text{def}}{=} b'(0)$ and $C \stackrel{\text{def}}{=} c'(0)$.

2.4°. Let us now analyze the differential equation (5). The equation (5) has two parameters B and C. They cannot be both equal to 0, since then (5) would imply that $D(t)$ is a constant, not depending on time t at all – but we know that the function $D(t)$ is decreasing. Thus, one of these two coefficients has to be different from 0. We therefore have three possible cases:

- the case when $B \neq 0$ and $C = 0$,
- the case when $B = 0$ and $C \neq 0$, and
- the case when $B \neq 0$ and $C \neq 0$.

Let us consider these three cases one by one.

2.5°. Let us first consider the case when $B \neq 0$ and $C = 0$. In this case, the equation (5) has the form $\frac{dD}{dt} = B \cdot D$. Moving all the terms containing D to the left-hand side and all the other terms to the right-hand side, we conclude that $\frac{dD}{D} = B \cdot dt$. Integrating both sides, we get $\ln(D) = B \cdot t + C_0$ for some constant C_0. Thus, for $D(t) = \exp(\ln(D(t)))$, we get the formula

$$D(t) = \exp(C_0) \cdot \exp(B \cdot t).$$

From the condition $D(0) = 1$, we conclude that $\exp(C_0) = 1$ and thus, $D(t) = \exp(B \cdot t)$. From the requirement that the function $D(t)$ be decreasing, we conclude that $B < 0$, i.e., that $B = -a$ for some $a > 0$, and $D(t) = \exp(-a \cdot t)$.

2.6°. Let us now consider the case when $B = 0$ and $C \neq 0$. In this case, the equation (5) has the form $\frac{dD}{dt} = -C \cdot D^2$. Moving all the terms containing D to the left-hand side and all the other terms to the right-hand side, we conclude that $\frac{dD}{D^2} = -C \cdot dt$. Integrating both sides, we get $-\frac{1}{D} = -C \cdot t + C_0$ for some constant C_0. Thus, $D(t) = \frac{1}{-C_0 + C \cdot t}$. From the condition $D(0) = 1$, we conclude that $C_0 = -1$ and thus, $D(t) = \frac{1}{1 + C \cdot t}$. From the requirement that the function $D(t)$ be decreasing, we conclude that $C > 0$. This is the hyperbolic expression for $k = C$.

2.7°. Let us now consider the generic case when $B \neq 0$ and $C \neq 0$. In this case, the equation (5) has the form $\frac{dD}{dt} = B \cdot D - C \cdot D^2$, i.e., equivalently, that $\frac{dD}{dt} = -C \cdot (D^2 - r \cdot D)$, where we denoted $r \stackrel{\text{def}}{=} \frac{B}{C}$. Moving all the terms containing D to the left-hand side and all the other terms to the right-hand side, we conclude that

$$\frac{dD}{D^2 - r \cdot D} = -C \cdot dt. \tag{6}$$

Here, $D^2 - r \cdot D = D \cdot (D - r)$. We can therefore represent the fraction $\frac{1}{D^2 - r \cdot D}$ as a linear combination of the fractions $\frac{1}{D}$ and $\frac{1}{D - r}$:

$$\frac{1}{D^2 - r \cdot D} = \frac{C_1}{D} + \frac{C_2}{D - r}. \tag{7}$$

Multiplying both sides of the equality (7) by the denominator of the left-hand side, we conclude that $1 = C_1 \cdot (D - r) + C_2 \cdot D$, i.e., that $1 = -C_1 \cdot r + (C_1 + C_2) \cdot D$.

This equality must hold for all D, so for $D=0$, we get $1=-C_1\cdot r$ and $C_1=-\dfrac{1}{r}$ and for $D=1$, we get $C_2=-C_1$ and thus, $C_2=\dfrac{1}{r}$. Thus, the formula (6) takes the form

$$\frac{1}{r}\cdot\left(\frac{dD}{D-r}-\frac{dD}{D}\right)=-C\cdot dt.$$

Multiplying both sides by $r=\dfrac{B}{C}$ and taking into account that $r\cdot C=\dfrac{B}{C}\cdot C=B$, we conclude that

$$\frac{dD}{D-r}-\frac{dD}{D}=-B\cdot dt.$$

Integrating both sides, we get

$$\ln(D-r)-\ln(D)=-B\cdot t+C_0 \tag{8}$$

for some constant C_0. Here,

$$\ln(D-r)-\ln(D)=\ln\left(\frac{D-r}{D}\right)=\ln\left(1-\frac{r}{D}\right).$$

Thus, the formula (8) takes the form

$$\ln\left(1-\frac{r}{D}\right)=-B\cdot t+C_0.$$

Applying $\exp(x)$ to both sides, we conclude that

$$1-\frac{r}{D}=C'\cdot\exp(-B\cdot t),$$

where $C'\stackrel{\text{def}}{=}\exp(C_0)$. Thus,

$$\frac{r}{D}=1-C'\cdot\exp(-B\cdot t),$$

and

$$D(t)=\frac{r}{1-C'\cdot\exp(-B\cdot t)}.$$

For $t=0$, we get $1=D(0)=\dfrac{r}{1-C'}$, hence $r=1-C'$, and we have

$$D(t)=\frac{1-C'}{1-C'\cdot\exp(-B\cdot t)}. \tag{9}$$

Let us now analyze what happens in the limit when $t\to\infty$, The corresponding asymptotics depends on where $B>0$ or $B<0$. If $B>0$, we get $\exp(-B\cdot t)\to 0$, hence the expression (9) tends to $1-C'$. The fact that this limit is 0 implies that $1-C'=0$, but in this case, $D(t)$ would be identically 0, and this is not the case (e.g., $D(0) = 1 \neq 0$). Thus, $B < 0$, i.e., $B = -k$ for some $k > 0$, and hence,

$$D(t) = \frac{1 - C'}{1 - C' \cdot \exp(k \cdot t)}.$$

Here, we cannot have $0 < C' < 1$, because otherwise, we will have the value $t = \frac{1}{k} \cdot \ln\left(\frac{1}{C'}\right)$ for which the denominator is 0 and for which, therefore, $D(t)$ is not defined. We cannot have $C' = 0$ – then $D(t)$ would not depend on time t at all; similarly, we cannot have $C' = 1$. So, must have either $C' < 0$ or $C' > 1$. When $C' < 0$, i.e., when $C' = -a$ for some $a > 0$, we get the expression

$$D(t) = \frac{1 + a}{1 + a \cdot \exp(k \cdot t)}.$$

When $C' > 1$, i.e., $C' = 1 + a$ for some $a > 0$, we get

$$D(t) = \frac{a}{(a + 1) \cdot \exp(k \cdot t) - 1}.$$

The proposition is proven.

4 Conclusions

For any person, an award that will only be delivered in the future is less valuable that the same amount delivered right away. To take this difference into account, economists use a special *discounting function* $D(t)$, so that an amount A delivered t moments in the future is equivalent to the discounted amount $D(t) \cdot A$ delivered today.

From the theoretical viewpoint, it seems that a natural discounting function is exponential $D(t) = D^t$: it corresponds, e.g., to the idea that a person can deposit the current amount $D(t) \cdot A$ in the bank, an after time t, get the value $(1 + q)^t \cdot D(t) \cdot A$, where q is the bank's interest rate. It is reasonable to equate this amount to the original amount A, getting $D(t) = D^t$ for $D = \frac{1}{1 + q}$.

Empirical research of human behavior shows, however, that in practice, when people make decisions, they use a different discounting function $D(t) = \frac{1}{1 + k \cdot t}$. To the best of our knowledge, so far, there has not been any quantitative theoretical explanation of this empirical phenomenon.

In this paper, we provide such an explanation. In this explanation, we use natural symmetry ideas.

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