An Ancient Bankruptcy Solution Makes Economic Sense

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Abstract While econometrics is a reasonable recent discipline, quantitative solutions to economic problem have been proposed since the ancient times. In particular, solutions have been proposed for the bankruptcy problem: how to divide the assets between the claimants? One of the challenges of analyzing ancient solutions to economics problems is that these solutions are often presented not as a general algorithm, but as a sequence of examples. When there are only a few such example, it is often difficult to convincingly extract a general algorithm from them. This was the case, for example, for the supposedly fairness-motivated Talmudic solution to the bankruptcy problem: only in the mid 1980s, the Nobelist Robert Aumann succeeded in coming up with a convincing general algorithm explaining the original examples. What remained not so clear in Aumann's explanation is why namely this algorithm best reflects the corresponding idea of fairness. In this paper, we find a simple economic explanation for this algorithm.

1 The Bankruptcy Problem and Its Ancient Solution: An Introduction

The bankruptcy problem: reminder. When a person or a company cannot pay all its obligation, a bankruptcy is declared, and the available funds are distributed

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among the claimants. Since there is not enough money to give, to each claimant, what he/she is owed, claimants will get less than what they are owed. How much less? What is a fair way to divide the available funds between the claimants?

An ancient solution. The bankruptcy problem is known for many millennia, since money became available and people starting lending money to each other. Solutions to this problem have also been proposed for many millennia. One such ancient solution is described in the Talmud, an ancient commentary on the Jewish Bible [2]. Specifically, this solution is described in the Babylonian Talmud, in Ketubot 93a, Bava Metzia 2a, and Yevamot 38a. (This solution is actually about a more general problem of several contracts which cannot be all fully fulfilled.)

Like many ancient texts containing mathematics, the Talmud does not contain an explicit algorithm. Instead, it contains four examples illustrating the main idea. In the first three examples, the three parties are owed the following amounts:

- the first person is owed $d_1 = 100$ monetary units,
- the second person is owed $d_2 = 200$ monetary units, and
- the third person is owed $d_3 = 300$ monetary units:

$$d_1 = 100, \ d_2 = 200, \ d_3 = 300.$$

For three different available amounts E, the text describes the amounts e_1, e_2, and e_3 that each of the three person will get:

	$d_1 = 100$	$d_2 = 200$	$d_3 = 300$
E	e_1	e_2	e_3
100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
200	50	75	75
300	50	100	150

There is also a fourth example, formulated in a slightly different way – as the question of dividing a disputed garment. In the bankruptcy terms, it can be described as follows: the owed amounts are:

$$d_1 = 50, \ d_2 = 100.$$

The available amount E and the recommended division (e_1, e_2) are as follows:

	$d_1 = 50$	$d_2 = 100$
E	e_1	e_2
100	25	75

Example are here, but what is a general solution? There has been, historically, a big problem with this solution: in contract to many other ancient mathematical texts, where the general algorithm is very clear from the examples, in this particular case, the general algorithm was unknown until 1985. Actually, many researchers came up with algorithms that explained *some* of these examples – while claiming that the original ancient text must have contained some mistakes.

Mystery solved, algorithm is reconstructed. This problem intrigued Robert Aumann, later the Nobel Prize winner in Economics (2005). In his 1985 paper [1], Professor Aumann came up with a reasonable general algorithm that explains the ancient solution; see also [4, 8].

To explain this algorithm, we need to first start with the the case of two claimants. Without losing generality, let us assume that the first claimant has a smaller claim $d_1 \le d_2$.

Then, if the overall amount E is small – to be precise, smaller that d_1 – then this amount E is distributed equally between the claimants, so that each gets

$$e_1 = e_2 = \frac{E}{2}.$$

When the available amount E is between d_1 and d_2, i.e., when $d_1 \le E \le d_2$, then the first claimant receives $e_1 = \dfrac{d_1}{2}$, and the second claimant receives the remaining amount $e_2 = E - e_1$.

This policy continues until we reach the amount $E = d_2$, at which moment the first claimant receives the amount $d_1 = \dfrac{d_1}{2}$ and the second claimant received the amount $e_2 = d_2 - \dfrac{d_1}{2}$. At this moment, after receiving the money, both claimants lose the same amount of money: $d_1 - e_1 = d_2 - e_2 = \dfrac{d_1}{2}$.

Finally, when the overall amount is larger than d_2 (but smaller than the overall amount of debt $d_1 + d_2$), the money is distributed in such a way that the losses remain equal, i.e., that $d_1 - e_1 = d_2 - e_2$ and $e_1 + e_2 = E$. From these two conditions, we can find the corresponding claims:

$$e_1 = \frac{E + d_1 - d_2}{2}, \;\; e_2 = \frac{E - d_1 + d_2}{2}.$$

The division between three (or more) claimants is then explained as the one for which for every two claimants, the amounts given to them is distributed according to the above algorithm. This can be easily checked if we select, for each pair (i, j) only the overall amount $E_{ij} = e_i + e_j$ allocated to claimants from this pair. As a result, for the pairs $(1,2)$, $(2,3)$, and $(1,3)$, we get the following tables:

	$d_1 = 100$	$d_2 = 200$
E_{12}	e_1	e_2
$66\frac{2}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
125	50	75
150	50	100

	$d_2 = 200$	$d_3 = 300$
E_{23}	e_2	e_3
$66\frac{2}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
150	75	75
250	100	150

	$d_1 = 100$	$d_3 = 300$
E_{13}	e_1	e_3
100	$66\frac{2}{3}$	$33\frac{1}{3}$
125	50	75
200	50	150

Remaining problem. That the ancient algorithm has been reconstructed, great. We now know *what* the ancients proposed. However, based on the above description, it is still not clear *why* this solution to the bankruptcy problem was proposed.

The above solution sounds rather arbitrary. To be more precise, both idea of dividing the amount equally and dividing the losses equally make sense, but how do we combine these two ideas? And why in the region between $E = \min(d_1, d_2)$ and $E = \max(d_1, d_2)$ the claimant with the smallest claim always gets half of his/her claim while the second claimant gets more and more? How dow that fit with the Talmud's claim that the proposed division represents fairness?

What we do in this paper. In this paper, we propose an economics-based explanation for the above solution.

2 Analysis of the Problem

What is fair is not clear. At first glance, it may look like fairness means dividing the amount either equally. If everyone is equal, why should someone gets more than others?

However, this is not necessarily a fair division. Suppose that two folks start with an equal amount of 400 dollars. They both decided to invest some money in the biomedical company that promised to use this money to develop a new drug curing up-to-now un-curable disease. The first person invested \$200, the second invested \$300. After this, the first person has \$200 left and the second person has \$100 left.

The company went bankrupt, and only \$300 remains in its account. If we divide this mount equally, both investors will get back the same amount of \$150. As a result:

- the first person will have \$350 instead of the original \$400, while
- the second person will have \$250 instead of the original \$400.

So, the first person loses only \$50, while the second person loses three times more: \$150. So, the first person, who selfishly kept money to himself, gets more than the altruistic second person who invested more in a noble case: how is this fair?

How we understand fairness: let us divide equally, but with respect to what status quo point? If two people jointly find an amount of money, then fairness means that this amount should be divided equally. If two people jointly contributed to some expenses, fairness means that they should split the expenses equally.

In both cases, we have a natural status quo point $(\widetilde{e}_1, \widetilde{e}_2)$:

- in the first case, we take $(\widetilde{e}_1, \widetilde{e}_2) = (0, 0)$, and
- in the second case, we take $(\widetilde{e}_1, \widetilde{e}_2) = (d_1, d_2)$.

Any change from the status quo should be divided equally, i.e., we should have $e_1 - \widetilde{e}_1 = e_2 - \widetilde{e}_2$. So, to apply this idea to the bankruptcy problem, we need to decide what is the status quo point here.

Comment. The idea that the difference between the actual amount and the status quo point should be divided equally is not only natural and fair, it actually comes from the game-theoretic notion of bargaining solution proposed by another Nobelist John Nash; see, e.g., [6, 7].

What are possible ranges for the status quo point: example. Let us consider one of the above cases, when the first person is owed $d_1 = 100$ monetary units, the second person is owed $d_2 = 200$ units, and we have an amount $E_{12} = 125$ units to distribute between these two claimants.

Depending on how we distribute this amount, the first person may get different amounts. The best possible case for the first claimant is when he get all the money he is owed, i.e., $\overline{e}_1 = 100$ monetary units. The worst possible case for the first claimant is when all the money goes to the second person, and the first person gets nothing: $\underline{e}_1 = 0$. Thus, the status quo point for the first person is somewhere in the interval

$$[\underline{e}_1, \overline{e}_1] = [0, 100].$$

Similarly, the best possible case for the second person is when the second person gets all the money, i.e., when $\overline{e}_2 = 125$. The worst possible case for the second person is when the first claimant gets everything he is owed – i.e., all 100 units, and the second person gets the remaining amount of $\underline{e}_2 = 125 - 100 - 25$ units. Thus, the status quo point for the second person is somewhere in the interval

$$[\underline{e}_2, \overline{e}_2] = [25, 125].$$

Let us perform the same analysis in the general case.

What are possible ranges for the status quo point: general case. Without losing generality, let us assume that the 1st person is the one who is owed less, i.e., that $d_1 \le d_1$. We will consider three different cases:

- when the available amount E_{12} does not exceed d_1: $E_{12} \le d_1$;
- when the available amount E_{12} is between d_1 and d_2: $d_1 \le E_{12} \le E_2$, and
- when the available amount E_{12} exceeds d_2, i.e., $d_2 \le E_{12} \le d_1 + d_2$.

Let us consider these three cases one by one.

Case when the overall amount does not exceed the smallest claim. Let us first consider the case when $E_{12} \le d_1 \le d_2$. In this case, for the first person, the best possible case is when this person gets all the amount E_{12}: $\overline{e}_1 = E_{12}$. The worst possible case is when all the available money goes to the second claimant and the first person gets nothing: $\underline{e}_1 = 0$. So, for the first person, the range of possible gains is $[\underline{e}_1, \overline{e}_1] = [0, E_{12}]$.

For the second person, the best possible case is when this person gets all the amount E_{12}: $\overline{e}_2 = E_{12}$. The worst possible case is when all the available money goes to the first claimant and the second person gets nothing: $\underline{e}_2 = 0$. So, for the second person, the range of possible gains is $[\underline{e}_2, \overline{e}_2] = [0, E_{12}]$.

Case when the overall amount is in the between the smaller and the larger claims. Let us now consider the case when $d_1 \le E_{12} \le d_2$. In this case, for the first person, the best possible case is when this person gets all the amount it is owed: $\overline{e}_1 = d_1$. The worst possible case is when all the available money goes to the second claimant and the first person gets nothing: $\underline{e}_1 = 0$. So, for the first person, the range of possible gains is $[\underline{e}_1, \overline{e}_1] = [0, d_1]$.

For the second person, the best possible case is when this person gets all the amount E_{12}: $\overline{e}_2 = E_{12}$. The worst possible case is when the first claimant gets all the money he is owed (i.e., the amount d_1), and the second person only gets the remaining amount $\underline{e}_2 = E_{12} - d_1$. So, for the second person, the range of possible gains is $[\underline{e}_2, \overline{e}_2] = [E_{12} - d_1, E_{12}]$.

Case when the overall amount is larger than both claims. Let us now consider the case when $d_1 \le d_2 \le E_{12}$. In this case, for the first person, the best possible case is when this person gets all the amount it is owed: $\overline{e}_1 = d_1$. The worst possible case is when the second person gets all the money it is owed, and the first person only gets the remaining amount $\underline{e}_1 = E_{12} - d_2$. So, for the first person, the range of possible gains is $[\underline{e}_1, \overline{e}_1] = [E_{12} - d_2, d_1]$.

For the second person, the best possible case is when this person gets all the amount it is owed: $\overline{e}_2 = d_2$. The worst possible case is when the first claimant gets all the money he is owed (i.e., the amount d_1), and the second person only gets the remaining amount $\underline{e}_2 = E_{12} - d_1$. So, for the second person, the range of possible gains is $[\underline{e}_2, \overline{e}_2] = [E_{12} - d_1, d_2]$.

Which points of the corresponding intervals should we select? In all three cases, for both claimants, we have an *interval* of possible values of the resulting gain. On each of these intervals, we need to select a status quo point that corresponds to the equivalent cost of this interval uncertainty.

The problem of what is the fair cost $\overline{e}$ in the case of interval uncertainty $[\underline{e}, \overline{e}]$ has been handled by yet another Nobelist, Leo Hurwicz; see, e.g., [3, 5, 6]. Namely, he proposed to select the value

$$\widetilde{e} = \alpha \cdot \overline{e} + (1 - \alpha) \cdot \underline{e},$$

where the coefficient $\alpha \in [0, 1]$ describes the decision-maker's degree of optimism-pessimism:

- the value $\alpha = 1$ describes a perfect optimist, when the decision maker only takes into account the most optimistic (best possible) scenario;
- the value $\alpha = 0$ describes a complete pessimist, when the decision maker only takes into account the worst possible scenario; and
- the values α strictly between 0 and 1 describe a realistic decision maker, who takes into account both the best-case and the worst-case possibilities.

Let us see what will happen if we take one of these solutions as a status-quo point and consider a division fair if the differences between the gains e_i and the status quo are equal: $e_1 - \widetilde{e}_1 = e_2 - \widetilde{e}_2$.

3 No Matter What Our Level of Optimism, We Get Exactly the Ancient Solution

Three cases: reminder. We will now show that in all the cases, we get exactly the ancient solution – so we have a good economic explanation for this solution. To show this, let us consider all three possible cases:

- case when $E_{12} \le d_1 \le d_2$,
- case when $d_1 \le E_{12} \le d_2$, and
- case when $d_1 \le d_2 \le E_{12}$.

Case when the overall amount does not exceed the smallest claim: general formulas. In this case,

$$\widetilde{e}_1 = \alpha \cdot \overline{e}_1 + (1 - \alpha) \cdot \underline{e}_1 = \alpha \cdot E_{12} + (1 - \alpha) \cdot 0 = \alpha \cdot E_{12}$$

and similarly,

$$\widetilde{e}_2 = \alpha \cdot \overline{e}_2 + (1 - \alpha) \cdot \underline{e}_2 = \alpha \cdot E_{12} + (1 - \alpha) \cdot 0 = \alpha \cdot E_{12}.$$

Thus, the fairness condition $e_1 - \widetilde{e}_1 = e_2 - \widetilde{e}_2$ takes the form $e_1 - \alpha \cdot E_{12} = e_2 - \alpha \cdot E_{12}$, i.e., the form $e_1 = e_2$.

So, in this case, no matter what is the optimism-pessimism value α, we divide the available amount E_{12} equally between the claimants:

$$e_1 = e_2 = \frac{E_{12}}{2}.$$

This is exactly what the ancient solution recommends in this case.

Case when the overall amount does not exceed the smallest claim: example. Let us consider one of the above examples, when $d_1 = 100$, $d_2 = 200$, and $E_{12} = 66\frac{2}{3}$. In this case, the above formulas recommend a solution in which $e_1 = e_2 = 33\frac{1}{3}$.

For the optimistic case $\alpha = 1$, the status quo point is $\widetilde{e}_1 = \overline{e}_1 = 66\frac{2}{3}$ and $\widetilde{e}_2 = \overline{e}_1 = 66\frac{2}{3}$. Thus, the condition of fairness with respect to this optimistic status quo point is indeed satisfied: $e_1 - \widetilde{e}_1 = e_2 - \widetilde{e}_2 = -33\frac{1}{3}$.

Case when the overall amount is in the between the smaller and the larger claims: general formulas. In this case,

$$\widetilde{e}_1 = \alpha \cdot \overline{e}_1 + (1-\alpha) \cdot \underline{e}_1 = \alpha \cdot d_1 + (1-\alpha) \cdot 0 = \alpha \cdot d_1$$

and

$$\widetilde{e}_2 = \alpha \cdot \overline{e}_2 + (1-\alpha) \cdot \underline{e}_2 = \alpha \cdot E_{12} + (1-\alpha) \cdot (E_{12} - d_1) = E_{12} - (1-\alpha) \cdot d_1.$$

Thus, the fairness condition $e_1 - \widetilde{e}_1 = e_2 - \widetilde{e}_2$ takes the form

$$e_1 - \alpha \cdot d_1 = e_2 - E_{12} + (1-\alpha) \cdot d_1 = e_2 - E_{12} + d_1 - \alpha \cdot d_1.$$

Canceling the common term $-\alpha \cdot d_1$ on both sides, we get $e_1 = e_2 - E_{12} + d_1$. Substituting $e_2 = E - e_1$ into this formula, we conclude that $e_1 = E_{12} - e_1 - E_{12} + d_1$, i.e., $e_1 = -e_1 + d_1$. Moving the term $-e_1$ to the left-hand side, we get $2e_1 = d_1$ and $e_1 = \frac{d_1}{2}$. The second person gets the remaining amount $e_2 = E_{12} - \frac{d_1}{2}$.

This is also exactly what the ancient solution recommends in this case.

Case when the overall amount is in the between the smaller and the larger claims: example. Let us consider one of the above examples, when $d_1 = 100$, $d_2 = 200$, and $E_{12} = 125$. In this case, the above formulas recommend a solution in which $e_1 = \frac{100}{2} = 50$ and $e_2 = E_{12} - e_1 = 125 - 50 = 75$.

Here, the optimistic status quo point is $\widetilde{e}_1 = d_1 = 100$ and $\widetilde{e}_2 = E_{12} = 125$. Thus, the condition of fairness with respect to this optimistic status quo point is indeed satisfied: $e_1 - \widetilde{e}_1 = 50 - 100 = -50$ and $e_2 - \widetilde{e}_2 = 75 - 125 = -50$.

Case when the overall amount is larger than both claims: general formulas. In this case,

$$\widetilde{e}_1 = \alpha \cdot \overline{e}_1 + (1-\alpha) \cdot \underline{e}_1 = \alpha \cdot d_1 + (1-\alpha) \cdot (E_{12} - d_2) =$$

$$\alpha \cdot d_1 + (1-\alpha) \cdot E_{12} - (1-\alpha) \cdot d_2$$

and

$$\widetilde{e}_2 = \alpha \cdot \overline{e}_2 + (1-\alpha) \cdot \underline{e}_2 = \alpha \cdot d_2 + (1-\alpha) \cdot (E_{12} - d_1) =$$

$$\alpha \cdot d_2 + (1-\alpha) \cdot E_{12} - (1-\alpha) \cdot d_1.$$

Thus, the fairness condition $e_1 - \widetilde{e}_1 = e_2 - \widetilde{e}_2$ takes the form

$$e_1 - \alpha \cdot d_1 - (1-\alpha) \cdot E_{12} + (1-\alpha) \cdot d_2 =$$

$$e_2 - \alpha \cdot d_2 - (1-\alpha) \cdot E_{12} + (1-\alpha) \cdot d_1.$$

Canceling the comon term $-(1-\alpha) \cdot E_{12}$ in both sides, we get

$$e_1 - \alpha \cdot d_1 + (1-\alpha) \cdot d_2 = e_2 - \alpha \cdot d_2 + (1-\alpha) \cdot d_1.$$

Moving terms containing d_1 and d_2 to the right-hand side, we conclude that $e_1 = e_2 + d_1 - d_2$. Substituting $e_2 = E_{12} - e_1$ into this formula, we get $e_1 = E_{12} - e_1 + d_1 - e_2$. Moving the term $-e_1$ to the left-hand side, we get $2e_1 = E_{12} + d_1 - e_2$ and $e_1 = \dfrac{E_{12} + d_1 - d_2}{2}$. The second person gets the remaining amount

$$e_2 = E_{12} - \frac{E_{12} + d_1 - d_2}{2} = \frac{E_{12} - d_1 + d_2}{2}.$$

This too is exactly what the ancient solution recommends in this case.

Case when the overall amount is larger than both claims: example. Let us consider one of the above examples, when $d_1 = 50$, $d_2 = 100$, and $E_{12} = 100$. In this case, the above formulas recommend a solution in which

$$e_1 = \frac{100 + 50 - 100}{2} = 25 \text{ and } e_2 = \frac{100 - 50 + 100}{2} = 75.$$

Here, the optimistic status quo point is $\widetilde{e}_1 = d_1 = 50$ and $\widetilde{e}_2 = d_2 = 100$. Thus, the condition of fairness with respect to this optimistic status quo point is indeed satisfied: $e_1 - \widetilde{e}_1 = 25 - 50 = -25$ and $e_2 - \widetilde{e}_2 = 75 - 100 = -25$.

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