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Fuzzy Analogues of Sets and Functions Can Be Uniquely Determined from the Corresponding Ordered Category: A Theorem

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Abstract: In modern mathematics, many concepts and ideas are described in terms of category theory. From this viewpoint, it is desirable to analyze what can be determined if, instead of the basic category of sets, we consider a similar category of fuzzy sets. In this paper, we describe a natural fuzzy analog of the category of sets and functions, and we show that, in this category, fuzzy relations (a natural fuzzy analogue of functions) can be determined in category terms – of course, modulo 1-1 mapping of the corresponding universe of discourse and 1-1 re-scaling of fuzzy degrees.

Keywords: fuzzy set; ordered category; category of fuzzy sets

1. Introduction

Category theory is one of the main tools of modern mathematics. Many mathematical theories can be naturally described in terms of a directed graph, where vertices are objects studied in this theory (e.g., sets in set theory, topological spaces in topology, linear spaces in linear algebra), and edges relate different objects: e.g., functions map one set into another, continuous mappings map one topological space into another one, linear mappings map one linear space into another one, etc. The corresponding graph is known as a *category*; see, e.g., [1].

In precise terms, a *category* is a tuple $(\text{Ob}, \text{Mor}, :, \text{id}, \circ)$, where:

- Ob is the set whose elements are called *objects*,
- Mor is a set whose elements are called *morphisms*,
- $:$ $\text{Mor} \rightarrow \text{Ob} \times \text{Ob}$ is a mapping that assigns, to each morphism $f \in \text{Mor}$ a pair of objects $(a, b) \in \text{Ob} \times \text{Ob}$; this is denoted by $f : a \rightarrow b$; the object a is called *f's domain*, and b is called *f's range*;
- id is a mapping that assigns, to each object $a \in \text{Ob}$, a morphism $\text{id}_a : a \rightarrow a$; and
- \circ is a mapping that assigns, to each pair of morphisms $f : a \rightarrow b$ and $g : b \rightarrow c$ for which the range of f is equal to the domain of g , a new morphism $g \circ f : a \rightarrow c$ so that for every $f : a \rightarrow b$, we have $\text{id}_b \circ f = f \circ \text{id}_a = f$.

Because of its universal character, category theory plays an important role in modern mathematics [1]. Many new mathematical concepts are defined in category terms, and many original concepts are re-formulated in category terms – such a reformulation in very general terms often enables mathematicians to generalize their ideas and results to a more general context.

As we have mentioned, different areas of mathematics can be described in terms of different categories:

- 31 • Set theory is naturally described in terms of a category *Set* in which objects are sets and
32 morphisms are functions.
- 33 • Topology is described in terms of a category *Top* in which objects are topological spaces and
34 morphisms are continuous mappings.
- 35 • Linear algebra is naturally described in terms of a category *Lin*, in which objects are linear spaces,
36 and morphisms are linear mappings, etc.

37 Many mathematical concepts can be reformulated in terms of an appropriate category; see, e.g.,
38 [3–5,8–14] and references therein.

39 **What happens in the fuzzy case?** If we allow fuzzy sets (see, e.g., see, e.g., [2,7,16,17,19]), what is
40 a natural analog of the category *Set*? In the category *Set*, morphisms from a to b are functions. In
41 the crisp case, for each function $f : a \rightarrow b$ and for each element $x \in a$, we have a unique value of
42 $y = f(x) \in b$.

43 Fuzzy means that for each $x \in a$, instead of a single value $y = f(x) \in b$, we may have different
44 possible values $y \in b$, with different degrees of confidence. In general, we can have all possible values
45 $y \in b$. For each $x \in a$ and for each $y \in b$, we have a degree $R_f(x, y) \in [0, 1]$ to which y is a possible
46 value of $f(x)$. Thus, a natural fuzzy analog of a function is a fuzzy relation.

Composition $g \circ f$ of fuzzy relations $f : a \rightarrow b$ and $g : b \rightarrow c$ can be defined in the usual way.
Namely, we want to know, for each pair of elements $x \in a$ and $c \in c$, to what extent there exists a $y \in b$
for which f brings us from a to b and g brings us from y to c . If we interpret “and” as \min and there
exists (an infinite “or”) as \max , then the above description translates into the following formula:

$$R_{g \circ f}(x, z) = \max_y \min(R_f(x, y), R_g(y, z)). \quad (1)$$

47 Since we have fuzzy relations, there is no need to explicitly describe the domain of each morphism:
48 if for some $x \notin a$, the value $f(x)$ is not defined, this simply means that for this x , we have $R_f(x, y) = 0$
49 for all $y \in b$. Similarly, there is no need to describe the range,

50 Thus, without losing generality, we can assume that we have only one object – the universal set U ,
51 and that the relation $R_f(x, y)$ is defined for all $x \in U$ and $y \in U$. Morphisms are then fuzzy relations,
52 with the usual composition relation (1).

53 **Need for an ordered category.** In the crisp case, every property is either true or false.

54 As we gain more information, we may get more confident in our knowledge. For example, we
55 may start with the situation in which, for a given x , several different values $f(x)$ are possible, but after
56 acquiring new information, we are becoming more and more confident that there is only one possible
57 value y_0 of $f(x)$. This means that for the remaining value y_0 , the degree of possibility $R_f(x, y_0)$ remains
58 the same, but for all $y \neq y_0$, the corresponding degree $R_f(x, y)$ decreases. To capture this phenomenon,
59 it is reasonable to supplement the category structure with the corresponding component-wise ordering
60 between fuzzy relations (morphisms): $f \leq f'$ if and only if $R_f(x, y) \leq R_{f'}(x, y)$ for all x and y .

61 **Formulation of the problem.** What can be defined based on this category-theory formulation? Can
62 we uniquely determine the elements of the Universe of discourse U and the corresponding relations
63 based on the categorical information?

64 **What we do in this paper.** In this paper, as an answer to the above questions, we present an axiomatic
65 description of fuzzy sets in the language of categories, with a proof of the soundness of this description.

66 2. Results

67 **Towards a precise formulation of the problem.** It is easy to see that if we have a 1-1 mapping
68 $\pi : U \rightarrow U$ of the Universe of discourse U onto itself (i.e., a bijection), then the corresponding
69 transformation $R(x, y) \rightarrow R(\pi(x), \pi(y))$ is an *automorphism* of the corresponding category in the sense
70 that it preserves the identity, composition, and order.

71 Similarly, if we have a 1-1 monotonically increasing mapping $H : [0,1] \rightarrow [0,1]$, then the
 72 transformation $R(x,y) \rightarrow H(R(x,y))$ is also such an automorphism. Indeed, since we only consider
 73 order between degrees, monotonic transformation of degrees should not change anything.

74 It turns out that modulo this simple equivalence, we can uniquely determine all the elements $x \in$
 75 U and all the relations $R(x,y)$ from the ordered category, i.e., in precise terms, that every automorphism
 76 is a composition of the automorphisms of the above two types. The proof of this result will be based
 77 on an explicit description of elements of U and relations $R_f(x,y)$ in category terms.

78 Let us describe the problem in precise terms.

79 **Definition 1.** By an ordered category, we mean a category in which for every two objects a and b , there is a
 80 partial order \leq on the set $\text{Mor}(a,b)$ of all morphisms from a to b .

81 **Definition 2.** Let U be a set; we will call it the Universe of discourse. By a U -fuzzy ordered category, we
 82 mean an ordered category in which:

- 83 • the only object is the set U ,
- 84 • morphisms are fuzzy relations, i.e., mappings $R : U \times U \rightarrow [0,1]$,
- 85 • the morphism id is defined as the mapping for which $\text{id}(x,x) = 1$ and $\text{id}(x,y) = 0$ for $x \neq y$,
- the composition of morphisms is defined by the formula

$$(g \circ f)(x,z) = \max_y \min(f(x,y), g(y,z)),$$

86 and

- 87 • the order between the morphisms is the component-wise order: $f \leq g$ means that $f(x,y) \leq g(x,y)$ for all
 88 x and y .

89 The U -fuzzy ordered category will be denoted by F_U .

90 *Comment.* One can easily see that this is indeed a category, i.e., that the composition of morphisms
 91 is associative, and the composition of any morphism f with the identity morphism id is equal to f :
 92 $f \circ \text{id} = \text{id} \circ f = f$.

93 **Definition 3.** An automorphism of an ordered category is a pair consisting of bijections $F : \text{Ob} \rightarrow \text{Ob}$ and
 94 $G : \text{Mor} \rightarrow \text{Mor}$ for which:

- 95 • for all f, a , and b , we have $f : a \rightarrow b$ if and only if $G(f) : F(a) \rightarrow F(b)$;
- 96 • for all f and g , we have $G(f \circ g) = G(f) \circ G(g)$,
- 97 • for all a , we have $G(\text{id}_a) = \text{id}_{F(a)}$, and
- 98 • for all f and g , we have $f \leq g$ if and only if $G(f) \leq G(g)$.

99 *Comment.* This definition is a natural generalization of the standard definition of automorphism of
 100 categories (see, e.g., [6,15,18]) to ordered categories.

101 **Proposition.** Let $\pi : U \rightarrow U$ be a bijection of U , and let $H : [0,1] \rightarrow [0,1]$ be an increasing bijection of the
 102 interval $[0,1]$. Then, the mapping $G_{\pi,H}$ that maps each morphism $f(x,y)$ into a morphism $(G_{\pi,H}(f))(x,y) =$
 103 $H(f(\pi(x), \pi(y)))$ is an automorphism of the category F_U .

104 Our main result is that these are the only automorphisms of the category F_U .

105 **Theorem.** For every set U , every automorphism of the ordered category F_U has the form $G_{\pi,H}$ for
 106 some bijection $\pi : U \rightarrow U$ and for some monotonic bijection $H : [0,1] \rightarrow [0,1]$.

107 *Comment.* This may not be very clear from the formulation of the result, but the proof will show that
 108 we can determine elements of the set U and values of the mappings $f(x,y)$ in category terms, i.e., we
 109 can indeed define fuzzy relations – a natural fuzzy analogue of functions – in category terms.

110 3. Proofs

111 3.1. Proof of the Proposition

112 This proposition is easy to prove: a permutation π does not change anything, and the increasing
113 bijection does not change the order.

114 3.2. Proof of the Theorem

115 1°. First, we can describe the morphism f_0 for which $f_0(x, y) = 0$ for all x and y in ordered-category
116 terms, as the only morphism f for which $f \leq g$ for all morphisms g .

117 Indeed, clearly $f_0 \leq g$ for all g . Vice versa, if $f \leq g$ for all g , then, in particular, $f \leq f_0$, i.e.,
118 $f(x, y) \leq f_0(x, y) = 0$ for all x and y , and since $f(x, y) \in [0, 1]$, this means that indeed $f(x, y) = 0$ for
119 all x and y .

120 2°. Let us first characterize all the morphisms $f \neq f_0$ for which the set $\{g : g \leq f\}$ is linearly ordered.
121 Since an automorphism preserves order, every automorphism maps such morphisms into morphisms
122 with the same property.

123 Specifically, we will prove that a morphism has this property if and only if we have $f(x, y) > 0$
124 only for one pair (x, y) , and we have $f(x', y') = 0$ for all other pairs (x', y') .

125 Indeed, one can easily check that for such morphisms f , the only morphisms $g \leq f$ are the
126 morphisms which also have $g(x', y') = 0$ for all pairs $(x', y') \neq (x, y)$. Such morphisms g are uniquely
127 described by the corresponding value $g(x, y)$. For every two such morphisms g and g' , depending on
128 whether $g(x, y) \leq g'(x, y)$ or $g'(x, y) \leq g(x, y)$, we have $g \leq g'$ or $g' \leq g$, i.e., the set $\{g : g \leq f\}$ is
129 indeed linearly ordered.

130 Vice versa, let us prove that if a morphism has this property, then it has $f(x, y) > 0$ only for one
131 pairs (x, y) . Indeed, if we have $f(x, y) > 0$ and $f(x', y') > 0$ for two different pairs $(x, y) \neq (x', y')$,
132 then we would be able to construct two different morphisms $g \leq f$ and $g' \leq f$ for which $g \not\leq g'$ and
133 $g' \not\leq g$. Namely, we take:

- 134 • $g(x, y) = f(x, y) > 0$ and $g(x'', y'') = 0$ for all pairs $(x'', y'') \neq (x, y)$, and
- 135 • $g'(x, y) = f(x', y') > 0$ and $g'(x'', y'') = 0$ for all pairs $(x'', y'') \neq (x', y')$.

136 This contradicts our assumption that the set $\{g : g \leq f\}$ is linearly ordered.

137 3°. Let us now describe, in ordered-category terms, morphisms f for which $f(x, x) > 0$ for some $a \in U$
138 and $f(x', y')$ for all other pairs $(x', y') \neq (x, x)$.

139 Indeed, out of all morphisms described in Part 2 of this proof, such morphisms can be determined
140 by the additional condition that $f \circ f = f$. This condition is clearly satisfied for such morphisms,
141 while for morphisms for which $f(x, y) > 0$ for some $b \neq a$, the composition $f \circ f$ is, as one can see,
142 identically 0 and thus, different from f .

143 4°. One can see that two morphisms f and f' of the type described in Part 3 are connected by the
144 relation \leq (i.e., $f \leq f'$ or $f' \leq f$) if and only if they correspond to the same element $a \in U$.

145 Thus, we can describe elements of the set U in ordered-category terms: as equivalent classes of
146 morphisms of the type described in Part 3 with respect to the relation $(f \leq f') \vee (f' \leq f)$.

147 Hence, if we have an automorphism, elements are mapped into elements in a 1-1 way, i.e., indeed
148 we have a bijection of the Universe of discourse.

149 5°. Let us now show that the degrees from the interval $[0, 1]$ can also be described – modulo increasing
150 bijections of this interval – in ordered-category terms.

151 5.1°. Indeed, for each element $a \in U$, different degrees $v \in [0, 1]$ can be associated with different
152 morphisms f described in Part 3 of this proof, i.e., morphisms for which:

- 153 • $f(x, x) > 0$ for this element a and

154 • $f(x', y')$ for all pairs $(x', y') \neq (x, x)$.

155 Different degrees are then simply associated with different values $v = f(x, x)$.

156 This construction provides us with degrees at each element $a \in U$. To get a general description of
157 degrees, we need to relate the values corresponding to different elements $x, x' \in U$.

158 5.2°. Let us denote, by $f_{x,v}$, the morphism for which:

- 159 • $f_{x,v}(x, x) = v$ and
160 • $f_v(x', y') = 0$ for all pairs $(x', y') \neq (x, x)$.

We want, for every $a \neq b$, to connect the values v and w corresponding to functions $f_{x,v}$ and $f_{y,w}$. This connection comes from the following auxiliary result:

$$w \leq v \Leftrightarrow \exists f_{x \rightarrow y} \exists f_{y \rightarrow x} (f_{x \rightarrow y} \circ f_{x,v} \cdot f_{y \rightarrow x} = f_{y,w}).$$

161 Indeed, by definition of a composition, the values of the composition $g \circ f$ cannot exceed the largest
162 value of each of the composed relations g and f . Thus, if $f_{x \rightarrow y} \circ f_{x,v} \cdot f_{y \rightarrow x} = f_{y,w}$, then the value
163 $f_{y,w}(b, b) = w$ cannot exceed the maximum value v of the function $f_{x,v}$; thus, $w \leq v$.

164 Vice versa, if $w \leq v$, then we can take the following morphisms $f_{x \rightarrow y}$ and $f_{y \rightarrow x}$:

- 165 • $f_{x \rightarrow y}(x, y) = w$ and $f_{x \rightarrow y}(x', y') = 0$ for all other pairs $(x', y') \neq (x, y)$, and, similarly,
166 • $f_{y \rightarrow x}(y, x) = w$ and $f_{y \rightarrow x}(x', y') = 0$ for all other pairs $(x', y') \neq (y, x)$.

167 In this case, as one can easily check, we have $f_{x \rightarrow y} \circ f_{x,v} \cdot f_{y \rightarrow x} = f_{y,w}$.

5.3°. Now that we know how to describe the relation $w \leq v$ for functions $f_{x,v}$ and $f_{y,w}$ in ordered-category form, we can describe equality $v = w$ between the degrees v and w corresponding to morphisms $f_{x,v}$ and $f_{y,w}$ as $(v \leq w) \& (w \leq v)$, i.e., in view of Part 5.2, as:

$$(\exists f_{x \rightarrow y} \exists f_{y \rightarrow x} (f_{x \rightarrow y} \circ f_{x,v} \cdot f_{y \rightarrow x} = f_{y,w})) \& (\exists g_{y \rightarrow x} \exists g_{x \rightarrow y} (g_{y \rightarrow x} \circ f_{y,w} \cdot g_{x \rightarrow y} = f_{x,v})).$$

168 This enables us to identify degrees $v \in [0, 1]$ in ordered-category terms – by identifying them
169 with the functions $f_{x,v}$ and taking into account the above possibility to compare degrees at different
170 elements a .

171 Hence, if we have an automorphism, degrees are mapped into degrees in a 1-1 and
172 order-preserving way, i.e., indeed we have a monotonic bijection $H : [0, 1] \rightarrow [0, 1]$.

173 6°. To complete the proof, we need to show how, for each morphism f and for every two elements a
174 and b , we can describe the value $f(x, y)$ in ordered-category terms. This will complete the proof that
175 the given automorphism has the form $G_{\pi, H}$ for the mappings π and H as identified in Sections 4 and 5
176 of this proof.

6.1°. Let us first prove the following auxiliary result:

$$\exists f_{y \rightarrow x} (f_{y \rightarrow x} \circ f_{y,1} \circ f \circ f_{x,1} = f_{x,v}) \Leftrightarrow v \leq f(x, y).$$

177 Indeed, by definition of a composition, the composition $c \stackrel{\text{def}}{=} f \circ f_{x,1}$ has the following form:

- 178 • $c(x, y') = f(x, y')$ for all y' and
179 • $c(x', y') = 0$ for all y' and for all $x' \neq a$.

180 Similarly, the composition $c' \stackrel{\text{def}}{=} f_{y,1} \circ f \circ f_{x,1} = f_{y,1} \circ c$ has the following form:

- 181 • $c'(x, y) = f(x, y)$, and
182 • $c'(x', y') = 0$ for all other pairs $(x', y') \neq (x, y)$.

183 As we have argued in Part 5 of this proof, the value of a composition function cannot exceed the
 184 maximum value of each of the composed morphisms. Thus, for the composition $f_{y \rightarrow x} \circ f_{y,1} \circ f \circ f_{x,1} =$
 185 $f_{y \rightarrow x} \circ c'$, the maximum value cannot exceed the maximum value $f(x, y)$ of the morphism c' . Thus, if
 186 $f_{y \rightarrow x} \circ c' = f_{x,v}$, the maximum value v of the morphism $f_{x,v}$ cannot exceed $f(x, y)$: $v \leq f(x, y)$.

187 Vice versa, for every $v \leq f(x, y)$, we can construct a morphism $f_{y \rightarrow x}$ for which $f_{y \rightarrow x} \circ c' = f_{x,v}$:
 188 namely, we can take:

- 189 • $f_{y \rightarrow x}(y, x) = v$, and
- 190 • $f_{y \rightarrow x}(x', y') = 0$ for all pairs $(x', y') \neq (y, x)$.

191 One can easily check that in this case indeed $f_{y \rightarrow x} \circ c' = f_{x,v}$.

192 6.2°. For each morphism f and for every two elements a and b , we can identify the degree $f(x, y)$ as
 193 the largest degree v for which the inequality $v \leq f(x, y)$ holds.

194 Since, according to Part 6.1 of this proof, the inequality $v \leq f(x, y)$ can be described in
 195 ordered-category terms, we can thus conclude that the degree $f(x, y)$ can also be described in
 196 ordered-category terms.

197 The proposition is proven.

198 4. Conclusions

199 Many concepts of modern mathematics, starting from the basic notions of sets and functions,
 200 are described in terms of category theory. many other mathematical concepts can be reformulated
 201 in category terms. Due to the general nature of category theory, such a reformulation often helps to
 202 extend notions and results from one area to different areas of mathematics.

203 Because of this potential advantage, it is reasonable to ask whether similar *fuzzy* notions can also
 204 be described in category terms. In this paper, we show that fuzzy relations – i.e., fuzzy analogues of
 205 functions – can indeed be described in category terms. Specifically, we show that, in the corresponding
 206 fuzzy category, we can describe both:

- 207 • elements of the original universe of discourse (modulo a 1-1 permutation), and
- 208 • fuzzy degrees (modulo a 1-1 monotonic mapping from the interval $[0, 1]$ onto itself).

209 This result shows the soundness of our axiomatic description of fuzzy sets in the language of categories.

210 At this moment, what we have is a very theoretical paper. However, we hope that, similarly to
 211 how the reformulation of crisp notions in category terms can help generalize the corresponding results,
 212 our reformulation will help extend fuzzy results to more general situations – and thus, will facilitate
 213 future applications.

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