The Sums of $m_i \cdot v_i$ and $m_i \cdot v_i^2$ Are Preserved, Why Not Sum of $m_i \cdot v_i^3$: A Pedagogical Remark

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Abstract

Students studying physics sometimes ask a natural question: the momentum – sum of $m_i \cdot v_i$ – is preserved, the energy – one half of the sum of $m_i \cdot v_i^2$ – is preserved, why not sum of $m_i \cdot v_i^3$? In this paper, we give a simple answer to this question.

1 Formulation of the problem

Students studying physics sometimes ask a natural question:

- the momentum – sum of $m_i \cdot v_i$ – is preserved,
- the energy – one half of the sum of $m_i \cdot v_i^2$ – is preserved, so
- why not sum of $m_i \cdot v_i^3$?

In this paper, we give a simple pedagogical answer to this question.

2 Our explanation

To answer the above question, let us consider a simple 2-body 1-D mechanical problem. We have two small (point-like) solid objects with masses $m_1 = 2$ and $m_1 = 1$ on a straight line, the second object is to the left of the first one. Originally:

- the first object is at rest (its velocity is $v_1 = 0$), while
- the second object moves towards the first object with velocity $v_2 = 1$.

What will happen when the second object hits the first one?

The situation is invariant with respect to rotations around the line connecting the two objects, so the resulting trajectories will also be invariant – and thus, both objects will continue to move along the same line. The only remaining question is: what will be the new velocities v_1' and v_2'?

We know that the momentum is preserved, and we know that kinetic energy is preserved. Thus, we can conclude that

$$m_1 \cdot v_1' + m_2 \cdot v_2' = m_1 \cdot v_1 + m_2 \cdot v_2 \tag{1}$$

and

$$m_1 \cdot (v_1')^2 + m_2 \cdot (v_2')^2 = m_1 \cdot v_1^2 + m_2 \cdot v_2^2. \tag{2}$$

Substituting the known values $m_1 = 2$, $m_2 = 1$, $v_1 = 0$, and $v_2 = 1$ into these formulas, we conclude that

$$2v_1' + v_2' = 1 \tag{1a}$$

and

$$2(v_1')^2 + (v_2')^2 = 1. \tag{2a}$$

From (1a), we conclude that

$$v_2' = 1 - 2v_1'. \tag{3}$$

Substituting this expression into the formula (2a), we get

$$2(v_1')^2 + (1 - 2v_1')^2 = 1.$$

Opening the parentheses, we get

$$2(v_1')^2 + 1 - 4v_1' + 4(v_1')^2 = 1.$$

Subtracting 1 from both sides and bringing similar terms together, we get

$$6(v_1')^2 - 4v_1' = 0,$$

i.e., equivalently,

$$v_1' \cdot (6v_1' - 4) = 0.$$

Thus, we have two options:

- either $v_1' = 0$,
- or $6v_1' - 4 = 0$, so that $v_1' = \frac{2}{3}$.

In the first case, when $v_1' = 0$, from the formula (3), we conclude that $v_2' = 1$. This means that the the first object remains immobile, and the second object continue with the same speed – so that it somehow passed through the first object. For solid objects (not for ghosts), this is not possible.

Since the case $v_1' = 0$ is not physically possible, we are left with the second option $v_1' = \frac{2}{3}$. In this case, the formula (3) implies that

$$v_2' = 1 - 2v_1' = 1 - \frac{4}{3} = -\frac{1}{3}.$$

One can check that in this case, the equalities (1) and (2) – describing preservation of the sums of $m_i \cdot v_i$ and $m_i \cdot v_i^2$ – are satisfied, while the sum of $m_i \cdot v_i^3$ are *not* preserved: indeed, in this case,

$$m_1 \cdot v_1^3 + m_2 \cdot v_2^3 = 2 \cdot 0^3 + 1 \cdot 1^3 = 1,$$

while

$$m_1 \cdot (v_1')^3 + m_2 \cdot (v_2')^3 = 2 \cdot \left(\frac{2}{3}\right)^3 + \left(-\frac{1}{3}\right)^3 =$$

$$2 \cdot \frac{8}{37} - \frac{1}{27} = \frac{16}{27} - \frac{1}{27} = \frac{15}{27} \neq 1.$$

In this example, we use specific weights and velocities, to make computations simpler, but we could get the same conclusion by taking almost any combination of initial masses and velocities.

Conclusion: we cannot require that the sum of the terms $m_i \cdot v_i^3$ is preserved: this way, there would be no way to describe a simple collision.

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References

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