Type-2 Fuzzy Analysis Explains Ubiquity of Triangular and Trapezoid Membership Functions

Olga Kosheleva and Vladik Kreinovich
University of Texas at El Paso
500 W. Universoty
El Paso, Texas 79968, USA
olgak@utep.edu, vladik@utep.edu

Shahnaz Shahbazova
Azerbaijan Technical University
Baku, Azerbaijan
shahbazova@gmail.com

***Abstract*—In principle, we can have many different membership functions. Interestingly, however, in many practical applications, triangular and trapezoidal membership functions are the most efficient ones. In this paper, we use fuzzy approach to explain this empirical phenomenon.**

I. Introduction

Triangular and trapezoid membership functions are ubiquitous: why? One of the main ideas behind fuzzy logic is to represent an imprecise ("fuzzy") natural-language property P like "small" by its *membership function*, i.e., a function $\mu(x)$ that assigns, to each possible value x of the corresponding property, the degree $\mu(x) \in [0,1]$ this value satisfies the property P (e.g., to what extent x is small); see, e.g., [1], [2], [3], [5], [6], [9].

According to this definition, we can have many different membership functions. However, in many applications of fuzzy techniques, the simplest piece-wise linear membership functions – e.g., triangular and trapezoid ones – works very well.

Why?

What we do in this paper. In this paper, we use fuzzy techniques to analyze this question. As a result of this analysis, we show that indeed triangular and trapezoid membership functions are the most reasonable ones. Thus, our analysis explains the ubiquity of triangular and trapezoid membership functions.

II. Analysis of the Problem

How can we analyze the problem: need for a type-2 approach. Traditionally – e.g., in control applications – fuzzy logic is used to select a value of the corresponding quantity, e.g., the value of control u. To come up with such a value:

- first, we use the experts' rules to come up, for each possible control value u, with a degree $d(u)$ to which this control value is reasonable;
- then, we select one of the control values u – e.g., the one for which the degree of reasonableness is the largest:

$$d(u) \to \max_u .$$

In our problem, instead of selecting a single *value* u, we select the whole membership *function* $\mu(x)$. To use fuzzy techniques for selecting μ, we thus need to do the following:

- first, we need to use experts' rules to assign, to each possible membership function $\mu(x)$, a degree $d(\mu)$ to which this membership function is reasonable, and
- then, out of all possible members functions, we select the one which is the most reasonable, i.e., for which the degree of reasonableness $d(\mu)$ is the largest:

$$d(\mu) \to \max_\mu .$$

Let us follow this path.

Comment. Traditionally:

- situations in which we use fuzzy to reason about real values is known as *type-1* fuzzy; while
- situations in which we use fuzzy to reason about fuzzy is known as *type-2 fuzzy* approach; see, e.g., [3], [4].

From this viewpoint, what we plan to use is an example of the type-2 fuzzy approach.

Expert rules. First, we need to select expert rules. We consider the problem in its utmost generality, we want rules that will be applicable to all possible fuzzy properties. In this case, the only appropriate rule that comes to mind is the following natural natural-language rule:

- if x and x' are close,
- then $\mu(x)$ and $\mu(x')$ should be close.

This rule exemplifies the whole idea of fuzziness: instead of abruptly changing the degree of confidence from 0 to 1 as would happen if we consider crisp properties (like $x \ge 0$), we have a smooth transition from 0 to 1.

How can we formalize this expert rule? Since there are infinitely many possible values of x and x', the above rule consists of infinitely many implications – one implication for each pair (x, x'). Dealing with *infinitely many* rules is difficult. It is therefore desirable to try to limit ourselves to *finite* number of rules.

Such a limitation is indeed possible. Indeed, theoretically, we can consider all infinitely many possible values x. However, in practice, the values of any physical quantity are bounded: e.g., locations on the Earth are bounded by the Earth's diameter, speeds are limited by the speed of light, etc. Thus, it is reasonable to assume that all possible values x are within some interval $[\underline{x}, \overline{x}]$.

Second, we only know x and x' with a certain accuracy $\varepsilon > 0$. From this viewpoint, there is no need to consider all infinitely many values, it is sufficient to consider only values on the grid of width ε, i.e., values

$$x_0 = \underline{x}, x_1 = \underline{x} + \varepsilon, x_2 = \underline{x} + 2\varepsilon, \ldots, x_n = \underline{x} + n \cdot \varepsilon = \overline{x},$$

where we denoted $n \stackrel{\text{def}}{=} \dfrac{\overline{x} - \underline{x}}{\varepsilon}$. In view of this, it is sufficient to describe the values $\mu_i \stackrel{\text{def}}{=} \mu(x_i)$ of the desired membership function at the points $x_0, x_1, \ldots, x_n$. We will call these values *discrete (d-)membership function*.

For these values, it is sufficient to formulate the above "closeness" rule only for neighboring values μ_i and μ_{i+1}. To be more precise, this rule now takes the following form:

$$\text{For all } i, \mu_i \text{ is close to } \mu_{i+1},$$

i.e., in other words,

$$(\mu_1 \text{ is close to } \mu_2) \text{ and } \ldots \text{and } ((\mu_{n-1} \text{ is close to } \mu_n). \quad (1)$$

Formula (1) can be formalized according to the usual fuzzy methodology. Intuitively, closeness of two numbers x and x' is equivalent to the requirement that the absolute value $d = |x - x'|$ of their difference is small. Thus, to express closeness, we need to select a membership function $s(d)$ describing "small". The larger the difference, the less small it is, so it is reasonable to require that the membership function $s(d)$ be strictly decreasing – at least until it reaches value 0 for the differences d which are clearly *not* small.

Since n is usually large, and thus, $1/n$ is small, without losing generality, we can safely assume that the distance $1/n$ is small, i.e., that $s(1/n) > 0$.

In terms of the selected membership function $\mu_0(d)$, for each i, the degree to which μ_i is close to μ_{i+1} is equal to $s(|\mu_i - \mu_{i+1}|)$. To find the degree $d(\mu)$ to which a given d-membership function $\mu = (\mu_1, \ldots, \mu_n)$ is reasonable (i.e., satisfies the above experts' rule), we need to apply some "and"-operation (t-norm) $f_\&(a, b)$ to these degrees, and get

$$d(\mu) = f_\&(s(|\mu_0 - \mu_1|), \ldots, s(|\mu_{n-1} - \mu_n|)).$$

It is reasonable to consider the simplest "and"-operation $f_\&(a, b) = \min(a, b)$, then we get

$$d(\mu) = \min(s(|\mu_0 - \mu_1|), \ldots, s(|\mu_{n-1} - \mu_n|)). \quad (2)$$

Now, we are ready to formulate the problem in precise terms.

III. Definitions and the Main Results

Definition 1. *Let n be a positive integer, and let $s(d)$ be a function from non-negative numbers to $[0, 1]$ which is strictly increasing until it reaches 0 and for which $s(1/n) > 0$.*

- *By a* discrete (d-) membership function*, we mean a tuple $\mu = (\mu_0, \ldots, \mu_n)$.*
- *By a* degree of reasonableness $d(\mu)$ *of a d-membership function μ, we mean the value*
$$d(\mu) = \min(s(|\mu_0 - \mu_1|), \ldots, s(|\mu_{n-1} - \mu_n|)). \quad (2)$$
- *Let M be a class of d-membership functions. We say that a d-membership function $\mu_{\text{opt}} \in M$ is* the most reasonable *d-membership function from the class M if*
$$d(\mu_{\text{opt}}) = \sup_{\mu \in M} d(\mu).$$
- *Let $\mathcal{M}$ be a class of membership functions defined on an interval $[\underline{x}, \overline{x}]$. We say that a membership function $\mu(x) \in \mathcal{M}$ is* the most reasonable *membership function from the class $\mathcal{M}$ if for a sequence $n_k \to \infty$, the corresponding d-membership functions are the most reasonable.*

Proposition 1. *Among all dmembership functions for which $\mu_0 = 0$ and $\mu_n = 1$, the most reasonable d-membership function is $\mu_i = \dfrac{i}{n}$.*

Comments.

- Notice that our result does not depend on the selection of the membership function $s(d)$.
- For reader's convenience, all the proofs are presented in the special proofs section.

Corollary 1. *Among all membership functions on the interval $[\underline{x}, \overline{x}]$ for which $\mu(\underline{x}) = 0$ and $\mu(\overline{x}) = 1$, the most reasonable membership function is $\mu(x) = \dfrac{x - \underline{x}}{\overline{x} - \underline{x}}$.*

Comment. Thus, the most reasonable membership function is linear.

Proposition 2. *Among all d-membership functions for which $\mu_0 = 1$ and $\mu_n = 0$, the most reasonable d-membership function is $\mu_i = \dfrac{n - i}{n}$.*

Corollary 2. *Among all membership functions on the interval $[\underline{x}, \overline{x}]$ for which $\mu(\underline{x}) = 1$ and $\mu(\overline{x}) = 0$, the most reasonable membership function is $\mu(x) = \dfrac{\overline{x} - x}{\overline{x} - \underline{x}}$.*

Comment. Thus, here also, the most reasonable membership function is linear.

This explains ubiquity of trapezoid membership functions. Let us consider a property P like "medium", for which:

- the property P is absolutely true for all values x from some interval $[\underline{t}, \overline{t}]$, and
- the property P is absolutely false for all x outside a wider interval $[\underline{T}, \overline{T}]$

Such properties are common.

In terms of membership degrees, the above condition means that:

- $\mu(x) = 0$ for $x \le \underline{T}$,
- $\mu(x) = 1$ for $\underline{t} \le x \le \overline{t}$, and
- $\mu(x) = 0$ for $x \ge \overline{T}$.

On the intervals $[\underline{T}, \underline{t}]$ and $[\overline{t}, \overline{T}]$, we do not know the values of the membership function. On both these subintervals, it is reasonable to select the most reasonable membership function.

Definition 2. *We say that a d-membership function* $\mu = (\mu_0, \ldots, \mu_n)$ *is* normalized *if* $\mu_i = 1$ *for some* i.

Proposition 3. *Among all normalized d-membership functions for which* $\mu_0 = \mu_{2k} = 0$*, the most reasonable d-membership function is the following one:*

- $\mu_i = \dfrac{i}{k}$ *when* $i \le k$*, and*
- $\mu_i = \dfrac{2k - i}{k}$ *when* $i \ge k$*.*

Corollary 3. *Among all normalized membership functions on the interval* $[\underline{x}, \overline{x}]$ *with midpoint* $\widetilde{x}$*, for which* $\mu(\underline{x}) = \mu(\overline{x}) = 0$*, the most reasonable membership function is:*

$$\mu(x) = \frac{x - \widetilde{x}}{\widetilde{x} - \underline{x}} \text{ for } x \le \widetilde{x} \text{ and } \mu(x) = \frac{\overline{x} - x}{\overline{x} - \widetilde{x}} \text{ for } x \ge \widetilde{x}.$$

Comment. Thus, here also, the most reasonable membership function is a triangular one.

Discussion. How robust are these results? To answer this question, let us show that under two somewhat different approaches, trapezoid and linear membership functions are still the most reasonable ones.

IV. First Set of Auxiliary Results: What If We Use a Different "And"-Operation, e.g., Product?

Discussion. In the previous section, we used the min "and"-operation. What if we use a different "and"-operation – e.g., the algebraic product $f_\&(a, b) = a \cdot b$, an operation also proposed by L. Zadeh in his original paper?

In this case, the result depends, in general, on the selection of the membership function $s(d)$ for "small". All we know about "small" is that 0 is definitely absolutely small, and that there exists some value D which is definitely not small. This is one of the cases discussed in the previous section, so let us use the results of the previous section to select the most reasonable membership function for small:

$$s_0(d) = 1 - \frac{d}{D}$$

for $d \le D$ and $s_0(d) = 0$ for $d \ge D$. For this selection, we get the following results.

Definition 3. *Let* n *be a positive integer, and let* $D > 0$ *be a positive real number. Let* $s_0(d) = 1 - \dfrac{d}{D}$ *for* $d \le D$ *and* $s_0(d) = 0$ *for* $d \ge D$*.*

- *By a* product-based degree of reasonableness $d_0(\mu)$ *of a d-membership function* μ*, we mean the value*
$$d_0(\mu) = s_0(|\mu_0 - \mu_1|) \cdot \ldots \cdot s_0(|\mu_{n-1} - \mu_n|).$$
- *Let* M *be a class of d-membership functions. We say that a membership function* $\mu_{\rm opt} \in M$ *is* the most product-based reasonable *membership function from the class* M *if*
$$d_0(\mu_{\rm opt}) = \sup_{\mu \in M} d_0(\mu).$$
- *Let* $\mathcal{M}$ *be a class of membership functions defined on an interval* $[\underline{x}, \overline{x}]$*. We say that a membership function* $\mu(x) \in \mathcal{M}$ *is* the most product-based reasonable reasonable *membership function from the class* $\mathcal{M}$ *if for a sequence* $n_k \to \infty$*, the corresponding d-membership function are the most product-based reasonable.*

Proposition 4. *Among all d-membership functions for which* $\mu_0 = 0$ *and* $\mu_n = 1$*, the most product-based reasonable d-membership function is* $\mu_i = \dfrac{i}{n}$*.*

Corollary 4. *Among all membership functions on the interval* $[\underline{x}, \overline{x}]$ *for which* $\mu(\underline{x}) = 0$ *and* $\mu(\overline{x}) = 1$*, the most product-based reasonable membership function is* $\mu(x) = \dfrac{x - \underline{x}}{\overline{x} - \underline{x}}$*.*

Comment. Thus, the most reasonable membership function is linear.

Proposition 5. *Among all d-membership functions for which* $\mu_0 = 1$ *and* $\mu_n = 0$*, the most product-based reasonable d-membership function is* $\mu_i = \dfrac{n - i}{n}$*.*

Corollary 5. *Among all membership functions on the interval* $[\underline{x}, \overline{x}]$ *for which* $\mu(\underline{x}) = 1$ *and* $\mu(\overline{x}) = 0$*, the most product-based reasonable membership function is* $\mu(x) = \dfrac{\overline{x} - x}{\overline{x} - \underline{x}}$*.*

Comment. Thus, here also, the most reasonable membership function is linear. Similarly to the previous section, this explains the ubiquity of trapezoid membership functions.

Proposition 6. *Among all normalized membership functions for which* $\mu_0 = \mu_{2k} = 0$*, the most product-based reasonable d-membership function is the following one:*

- $\mu_i = \dfrac{i}{k}$ *when* $i \le k$*, and*
- $\mu_i = \dfrac{2k - i}{k}$ *when* $i \ge k$*.*

Corollary 6. *Among all normalized membership functions on the interval* $[\underline{x}, \overline{x}]$ *with midpoint* $\widetilde{x}$*, for which* $\mu(\underline{x}) = \mu(\overline{x}) = 0$*, the most product-based reasonable membership function is:*

$$\mu(x) = \frac{x - \widetilde{x}}{\widetilde{x} - \underline{x}} \text{ for } x \le \widetilde{x} \text{ and } \mu(x) = \frac{\overline{x} - x}{\overline{x} - \widetilde{x}} \text{ for } x \ge \widetilde{x}.$$

Comment. Thus, here also, the most reasonable membership function is a triangular one.

V. Second Set of Auxiliary Results: What If We Use Statistics-Motivated Least Squares Approach to Select the Most Reasonable Membership Function

Discussion. In the above sections, we used fuzzy techniques to determine the degree to which a d-membership function is reasonable, i.e., a degree to which $\mu_1 - \mu_0$ is small, $\mu_2 - \mu_1$ is small, etc. Intuitively, small means close to 0, i.e., being approximately equal to 0. In other words, we determine a degree to which the following system of approximate equalities hold:

$$\mu_1 - \mu_0 \approx 0, \ldots, \mu_n - \mu_{n-1} \approx 0.$$

It is worth noticing that such systems of approximate equation are well known in traditional statistical data analysis, where the usual way of dealing with such system is to use the Least Squares approach (see, e.g., [7], [8]), i.e., to look for the solutions for which the sum of the squares of the approximation errors is the smallest possible:

$$(\mu_1 - \mu_0)^2 + \ldots + (\mu_n - \mu_{n-1})^2 \to \min .$$

Thus, we arrive at the following definitions.

Definition 4. *Let n be a positive integer.*

- *By the* least-squares degree of reasonableness $d_1(\mu)$ *of a d-membership function μ, we mean the value*

$$d_1(\mu) = (\mu_0 - \mu_1)^2 + \ldots + (\mu_{n-1} - \mu_n)^2.$$

- *Let M be a class of d-membership functions. We say that a membership function $\mu_{\rm opt} \in M$ is* the most least-squares reasonable *membership function from the class M if*

$$d_1(\mu_{\rm opt}) = \sup_{\mu \in M} d_1(\mu).$$

- *Let $\mathcal{M}$ be a class of membership functions defined on an interval $[\underline{x}, \overline{x}]$. We say that a membership function $\mu(x) \in \mathcal{M}$ is* the most least-squares reasonable reasonable *membership function from the class $\mathcal{M}$ if for a sequence $n_k \to \infty$, the corresponding d-membership functions are the most least-squares reasonable.*

Proposition 7. *Among all d-membership functions for which $\mu_0 = 0$ and $\mu_n = 1$, the most least-squares reasonable d-membership function is $\mu_i = \dfrac{i}{n}$.*

Corollary 7. *Among all membership functions on the interval $[\underline{x}, \overline{x}]$ for which $\mu(\underline{x}) = 0$ and $\mu(\overline{x}) = 1$, the most least-squares reasonable membership function is $\mu(x) = \dfrac{x - \underline{x}}{\overline{x} - \underline{x}}$.*

Comment. Thus, the most reasonable membership function is linear.

Proposition 8. *Among all d-membership functions for which $\mu_0 = 1$ and $\mu_n = 0$, the most least-squares reasonable d-membership function is $\mu_i = \dfrac{n-i}{n}$.*

Corollary 8. *Among all membership functions on the interval $[\underline{x}, \overline{x}]$ for which $\mu(\underline{x}) = 1$ and $\mu(\overline{x}) = 0$, the most least-squares reasonable membership function is $\mu(x) = \dfrac{\overline{x} - x}{\overline{x} - \underline{x}}$.*

Comment. Thus, here also, the most reasonable membership function is linear. Similarly to the previous section, this explains the ubiquity of trapezoid membership functions.

Proposition 9. *Among all normalized membership functions for which $\mu_0 = \mu_{2k} = 0$, the most least-squares reasonable d-membership function is the following one:*

- $\mu_i = \dfrac{i}{k}$ *when $i \le k$, and*
- $\mu_i = \dfrac{2k-i}{k}$ *when $i \ge k$.*

Corollary 9. *Among all normalized membership functions on the interval $[\underline{x}, \overline{x}]$ with midpoint $\widetilde{x}$, for which $\mu(\underline{x}) = \mu(\overline{x}) = 0$, the most least-squares reasonable membership function is:*

$$\mu(x) = \frac{x - \widetilde{x}}{\widetilde{x} - \underline{x}} \text{ for } x \le \widetilde{x} \text{ and } \mu(x) = \frac{\overline{x} - x}{\overline{x} - \widetilde{x}} \text{ for } x \ge \widetilde{x}.$$

Comment. Thus, here also, the most reasonable membership function is a triangular one.

VI. Proofs

Proof of Proposition 1.

1°. Let us first prove, by contradiction, that for every d-membership function μ from the class M, we have

$$d(\mu) \le s_0 \left(\frac{1}{n}\right).$$

Indeed, if we had

$$d(\mu) > s(\frac{1}{n}),$$

then, by definition of the degree $d(\mu)$, this would means that

$$s(|\mu_i - \mu_{i+1}|) > s\left(\frac{1}{n}\right)$$

for all i. Since the function $s(d)$ is strictly decreasing, this implies that

$$|\mu_i - \mu_{i+1}| < \frac{1}{n}$$

for all i. However, we always have

$$|a + \ldots + b| \le |a| + \ldots + |b|.$$

Here,

$$\mu_0 - \mu_n = (\mu_0 - \mu_1) + \ldots + (\mu_{n-1} - \mu_n),$$

hence

$$|\mu_0 - \mu_n| \le |\mu_0 - \mu_1| + \ldots + |\mu_{n-1} - \mu_n|. \quad (3)$$

However, the left-hand side is equal to $|0 - 1| = 1$, while the right hand side is the sum of n terms each of which is smaller than $\dfrac{1}{n}$, hence the sum is smaller than 1. This contradiction shows that the case

$$d(\mu) > s\left(\frac{1}{n}\right)$$

is indeed impossible.

2°. One can easily check that for

$$x_i = \frac{i}{n},$$

we have

$$|\mu_i - \mu_{i+1}| = \frac{1}{n},$$

hence

$$s(|x_i - x_{i+1}|) = s\left(\frac{1}{n}\right)$$

for all i, and

$$d(\mu) = s\left(\frac{1}{n}\right).$$

3°. Let us prove that, vice versa, if

$$d(\mu) = s\left(\frac{1}{n}\right),$$

then

$$x_i = \frac{i}{n}$$

for all i.

Indeed, if

$$d(\mu) = s\left(\frac{1}{n}\right),$$

then for each i, we have

$$s(|\mu_i - \mu_{i+1}|) \geq s\left(\frac{1}{n}\right),$$

hence, due to strict monotonicity of the function $s(d)$, we have

$$|\mu_i - \mu_{i+1}| \leq \frac{1}{n}.$$

If one of the values $|\mu_i - \mu_{i+1}|$ was smaller than

$$\frac{1}{n},$$

then the sum

$$|\mu_0 - \mu_1| + \ldots + |\mu_{n-1} - \mu_n|$$

would be smaller than 1, which contradicts to the inequality (3). Similarly, a difference $\mu_{i+1} - \mu_i$ cannot be negative, since then the sum of all the values $\mu_{i+1} - \mu_i$, which is equal to $\mu_n - \mu_0 = 1$, would be smaller than 1.

Thus,

$$\mu_{i+1} - \mu_n = \frac{1}{n}$$

for all i, hence, for each i, we have:

$$\mu_i = \mu_0 + (\mu_1 - \mu_0) + \ldots + (\mu_i - \mu_{i-1}) =$$
$$\frac{1}{n} + \ldots + \frac{1}{n}(i \text{ times}) = \frac{i}{n}.$$

The proposition is proven.

Proof of Corollary 1. If we take into account that $\mu_i = \mu(x_i)$ and $x_i = \underline{x} + i \cdot \varepsilon$, we conclude that

$$i = \frac{x_i - \underline{x}}{\varepsilon}.$$

Substituting this expression for i into the formula

$$\mu(x_i) = \frac{i}{n},$$

we conclude that

$$\mu(x_i) = \frac{x_i - \underline{x}}{n \cdot \varepsilon}.$$

Here, by definition of n, we have $n \cdot \varepsilon = \overline{x} - \underline{x}$, hence

$$\mu(x_i) = \frac{x_i - \underline{x}}{\overline{x} - \underline{x}}.$$

The corollary is proven.

Proofs of Proposition 2 and Corollary 2 are similar to the proofs of Proposition 1 and Corollary 1.

Proof of Proposition 3. Let i_0 denote the value for which $\mu_{i_0} = 1$. Then, we have $d(\mu) = \min(d^-(\mu), d^+(\mu))$, where we denoted

$$d^-(\mu) = \min(s(|\mu_0 - \mu_1|), \ldots, s(|\mu_{i_0-1} - \mu_{i_0}|))$$

and

$$d^+(\mu) = \min(s(|\mu_{i_0} - \mu_{i_0+1}|), \ldots, s(|\mu_{n-1} - \mu_n|))$$

Similarly to the proof of Proposition 1, we can conclude that

$$d^-(\mu) \leq s\left(\frac{1}{i_0}\right)$$

and

$$d^+(\mu) \leq s\left(\frac{1}{n - i_0}\right).$$

If

$$i_0 < \frac{n}{2},$$

then

$$n - i_0 > \frac{n}{2},$$

hence

$$d(\mu) \leq d^+(\mu) \leq s\left(\frac{1}{n - i_0}\right) < s\left(\frac{2}{n}\right).$$

Similarly, if

$$i_0 > \frac{n}{2},$$

then

$$d(\mu) \leq d^-(\mu) \leq s\left(\frac{1}{i_0}\right) < s\left(\frac{2}{n}\right).$$

On the other hand, when

$$i_0 = \frac{n}{2}$$

and values μ_i are equidistant, we get

$$s(\mu) = s\left(\frac{2}{n}\right).$$

Thus, for the optimal d-membership function, we must have

$$i_0 = \frac{n}{2}.$$

In this case, similarly to the proofs of Propositions 1 and 2, we can conclude that the only way to get

$$d(\mu) = s\left(\frac{2}{n}\right)$$

is to have values μ_i uniformly changing on each of the intervals $[0, i_0]$ and $[i_0, n]$, i.e., to have

$$\mu_i = \frac{i}{k}$$

for

$$i \le k = \frac{n}{2}$$

and

$$\mu_i = \frac{n-i}{k}$$

for $i \ge k$. The proposition is proven.

Proof of Corollary 3 is similar to the proofs of Corollaries 1 and 2: we plug in the expression for i in terms of x_i into the formula for $\mu_i = \mu(x_i)$.

Proof of Proposition 4. For

$$\mu_i = \frac{i}{n},$$

we have

$$|\mu_i - \mu_{i+1}| = \frac{1}{n},$$

thus

$$s_0(|\mu_i - \mu_{i+1}|) = 1 - \frac{1}{n \cdot D}$$

and

$$d_1(\mu) = \left(1 - \frac{1}{n \cdot D}\right)^n.$$

Let us prove that for all other d-membership functions, we have smaller values of $d_0(\mu)$. Indeed, it is known that the geometric mean is always smaller than or equal than the arithmetic mean, and the only time when they are equal is when all the values are the same. For the values

$$s_0(|\mu_i - \mu_{i+1}|) = 1 - \frac{1}{n \cdot D},$$

the geometric mean is equal to $(d_0(\mu))^{1/n}$, while the arithmetic mean a is equal to

$$a = 1 - \frac{|\mu_0 - \mu_1| + \ldots + |\mu_{n-1} - \mu_n|}{n \cdot D}.$$

As we have proven earlier, the sum

$$|\mu_0 - \mu_1| + \ldots + |\mu_{n-1} - \mu_n|$$

is always greater than or equal to 1, and it is equal to 1 only if all the differences $\mu_{i+1} - \mu_i$ are positive. Thus, the arithmetic mean is always smaller than or equal to

$$1 - \frac{1}{n \cdot D}.$$

So, the fact that the geometric mean is smaller than or equal to the arithmetic mean implies that

$$(s_0(\mu))^{1/n} \le a \le 1 - \frac{1}{n \cdot D}.$$

Thus,

$$s_0(\mu) \le a^n \le \left(1 - \frac{1}{n \cdot D}\right)^n.$$

So, we have the desired strict inequality except for the case when all the differences $\mu_{i+1} - \mu_i$ are equal and positive. This is exactly the case of

$$\mu_i = \frac{i}{n}.$$

The proposition is proven.

Proofs of Propositions 5 and 6 and Corollaries 4–6 are similar.

Proof of Proposition 7. To find the values μ_i minimizing the sum of the squares, we can differentiate this sum with respect to μ_i and equate the derivative to 0. We then conclude that

$$(\mu_i - \mu_{i-1}) + (\mu_i - \mu_{i+1}) = 0,$$

i.e., that

$$\mu_i - \mu_{i-1} = \mu_{i+1} - \mu_i.$$

So, all the differences $\mu_{i+1} - \mu_i$ are equal to each other. Since their sum is equal to 1, each difference is equal to $\dfrac{1}{n}$, hence $\mu_i = \dfrac{i}{n}$. The proposition is proven.

Proofs of Propositions 8 and 9 and Corollaries 7–9 are similar.

ACKNOWLEDGMENTS

This work was supported in part by the US National Science Foundation grant HRD-1242122.

REFERENCES

[1] R. Belohlavek, J. W. Dauben, and G. J. Klir, *Fuzzy Logic and Mathematics: A Historical Perspective*, Oxford University Press, New York, 2017.

[2] G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall, Upper Saddle River, New Jersey, 1995.

[3] J. M. Mendel, *Uncertain Rule-Based Fuzzy Systems: Introduction and New Directions*, Springer, Cham, Switzerland, 2017.

[4] J. M. Mendel and D. Wu, *Perceptual Computing: Aiding People in Making Subjective Judgments*, IEEE Press and Wiley, New York, 2010.

[5] H. T. Nguyen and E. A. Walker, *A First Course in Fuzzy Logic*, Chapman and Hall/CRC, Boca Raton, Florida, 2006.

[6] V. Novák, I. Perfilieva, and J. Močkoř, *Mathematical Principles of Fuzzy Logic*, Kluwer, Boston, Dordrecht, 1999.

[7] S. G. Rabinovich, *Measurement Errors and Uncertainty: Theory and Practice*, Springer Verlag, Berlin, 2005.

[8] D. J. Sheskin, *Handbook of Parametric and Nonparametric Statistical Procedures*, Chapman and Hall/CRC, Boca Raton, Florida, 2011.

[9] L. A. Zadeh, "Fuzzy sets", *Information and Control*, 1965, Vol. 8, pp. 338–353.