Towards Foundations of Fuzzy Utility: Taking Fuzziness into Account Naturally Leads to Intuitionistic Fuzzy Degrees

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Abstract. The traditional utility-based decision making theory assumes that for every two alternatives, the user is either absolutely sure that the first alternative is better, or that the second alternative is better, or that the two alternatives are absolutely equivalent. In practice, when faced with alternatives of similar value, people are often not fully sure which of these alternatives is better. To describe different possible degrees of confidence, it is reasonable to use fuzzy logic techniques. In this paper, we show that, somewhat surprisingly, a reasonable fuzzy modification of the traditional utility elicitation procedure naturally leads to intuitionistic fuzzy degrees.

1 Formulation of the Problem

Need to help people make decisions. In many practical situations, we need to make a decision, i.e., we need to select an alternative which is, for us, better than all other possible alternatives.

If the set of alternatives is small, we can easily make such a decision: indeed, we can easily compare each alternative with every other one, and, based on these comparisons, decide which one is better. However, when the number of alternatives becomes large, we have trouble making decisions. Even in simple situations, when we are looking for cereal in a supermarket, there are usually so many selections that we just ignore most of them and go with a familiar one – instead of the optimal one.

The situation is even more complicated if we are trying to make a decision not on behalf of ourselves, but rather on behalf of a company or a community. In this case, even comparing two alternatives is not easy: it requires taking into account interests of different people involved, so the decision making process becomes even more complicated.

Traditional approach to decision making: the notion of utility. The traditional approach to decision making was originally motivated by the idea of money.

We all know what money is, but when money was invented, it was a revolutionary idea that made economic exchange much easier. Indeed, before money was invented, people exchanged goods by barter: chicken for a shirt, jewelry for boots, etc. Thus, to make a proper decision, every person needed to be able to compare every two items with each other: how many chickens is this person willing to exchange for a shirt, how many boots for a golden earing, etc. For n goods, we have $\frac{n \cdot (n-1)}{2} \approx \frac{n^2}{2}$ possible pairs. So, each person had to have in mind a table of $n^2/2$ numbers.

With money as a universally accepted means of exchange, all the person needs to do is to decide, for each of n items, how much he or she is willing to pay for 1 unit. So, to successfully make decisions, it is sufficient to know n numbers – the values of each of n items. Then, even when we want to barter, we can easily decide how many chickens are worth a shirt: it is sufficient to divide the price of a shirt by the price of a chicken.

A similar idea can be used to compare different alternatives. All we need is to have a numerical scale, i.e., a 1-parametric family of "standard" alternatives whose quality increases with the increase in the value of the parameter. This can be the money amount. Alternatively, this can be the probability p of a lottery in which we something very good: the larger the probability, the more preferable the lottery.

Then, instead of comparing every alternative with every other alternative, we simply compare every alternative with alternatives on the selected scale, and thus, for each alternative, we find the numerical value of the standard alternative which is equivalent to a given one. This numerical value is known as the *utility* $u(a)$ of a given alternative a; see, e.g., [3, 4, 6, 8, 11].

In terms of utility, an alternative a is better than the alternative a' if and only the utility $u(a)$ of the alternative a is larger than the utility $u(a')$ of the alternative a'. Thus, once we have found the utility $u(a)$ of each alternative, then it is easy to predict which alternative the person will select: he/she will select the alternative for which the utility $u(a)$ is the largest possible.

How to actually find the utility. From the algorithmic viewpoint, the fastest way to find the utility of a given alternative a based on binary comparisons is to use bisection. Usually, we have an a prior lower bound and an a priori upper bound for the desired utility $u(a)$: $\underline{u} \le u(a) \le \overline{u}$. In other words, we know that the desired utility $u(a)$ is somewhere in the interval $[\underline{u}, \overline{u}]$. In this procedure, we will narrow down this interval.

Once an interval is given, we can compute its midpoint $\widetilde{u} = \frac{\underline{u} + \overline{u}}{2}$ and compare a with the corresponding standard alternative $s(\widetilde{u})$.

If a is exactly equivalent to the resulting standard alternative, this means that we have found the exact value of the utility $u(a)$: it is equal to $\widetilde{u}$. However, such exact equivalences are rare; in most cases, we will find out that:

– either a is better than $s(\widetilde{u})$; we will denote it by $s(\widetilde{u}) < a$; or
– the standard alternative is better: $a < s(\widetilde{u})$.

In the first case, the preference $s(\widetilde{u}) < a$ means that $\widetilde{u} < u(a)$. Thus, we know that $u(a) \in [\widetilde{u}, \overline{u}]$. In other words, we have a new interval containing the desired utility. We can obtain this new interval if we replace the previous lower bound $\underline{u}$ with the new lower bound $\widetilde{u}$.

In the second case, the preference $a < s(\widetilde{u})$ means that $u(a) < \widetilde{u}$. Thus, we know that $u(a) \in [\underline{u}, \widetilde{u}]$. In other words, we have a new interval containing the desired utility. We can obtain this new interval if we replace the previous upper bound $\overline{u}$ with the new upper bound $\widetilde{u}$.

In both cases, the width of the interval is decreased by a factor of 2. Then, we can repeat this procedure, and in k steps, we get $u(a)$ with accuracy 2^{-k}. For example, in 7 steps, we get an accuracy of 1%.

Need to take fuzziness into account. The above procedure works well if a person is absolutely sure about his/her preferences. In practice, we are often not 100% sure about our preferences, especially when we compare alternatives of nearby value.

It is reasonable to describe this uncertainty in fuzzy terms. For example, if we use money as a standard scale, then for each alternative a, instead of having a single amounts of money equivalent to this item, we may have different amounts with different degree of certainty. In other words, instead of the above crisp model, in which a person has an exact utility value $u(a)$ for each alternative a, we know have a fuzzy model in which for each person and for each alternative a, we have a membership function $\mu_a(u)$ that describes, for each possible value u, to what extend this value u is equivalent to the alternative a; see, e.g., [2, 5, 7, 9, 10, 12].

How to elicit fuzzy utility: a reasonable idea. We know how to elicit crisp utility $u(a)$ of a given alternative a: we need to compare the alternative a with different values u_0 of the scale. In the case of fuzzy utility, it is reasonable to apply the same procedure. The only difference is that now, since the utility value $u(a)$ is fuzzy, this comparison will not lead to a crisp "yes"-"no" answer; instead, we will get a fuzzy answer – the degree to which it is possible that a is better than u_0 (and, if needed, the degree to which it is possible that a is worse than u_0).

Remaining open problems and what we do in this paper. In the crisp case, we can determine the utility value $u(a)$ from the results of the user's comparisons.

To deal with the more realistic fuzzy case, we need to be able to extract the fuzzy utility from the fuzzy answers to different comparisons. This is the question that we deal with in this paper.

Interestingly, it turns out that in this context, intuitionistic fuzzy degrees (see, e.g., [1]) naturally appear – in other words, instead of a single degree of confidence in each corresponding statement, we now get *two* degrees:

– the degree to which this statement is true, and

– the degree to which this statement is false,

and, in contrast to the traditional fuzzy logic, these degrees do not add up to 1.

2 Analysis of the Problem

What happens if we compare the alternative a with a fixed value u_0 on the utility scale? As we have mentioned earlier, while in the crisp case, each alternative a is equivalent to a single number $u(a)$ on the utility scale, in general, the utility of an alternative is characterized not by a single number, but rather with a membership function $\mu_a(u)$. This function describes, for each value u from the utility scale, to what extent the alternative a is equivalent to u.

What will happen is we compare the alternative a to a value u_0 on the utility scale? In the crisp case, since the changes that a is exactly equivalent to a_0 are slim, we have either $a < u_0$ or $u_0 < a$. So, we can ask whether a is better than u_0, or we can ask whether u_0 is better than a – whatever question we ask, we get the exact same information.

Let us first consider the question of whether a is better than u_0, i.e., whether $u_0 < a$. How can we extend this to the fuzzy case? To perform this extension, it is convenient to take into account that while from the purely mathematical viewpoint, $<$ is a relation – and in mathematics, relations usually treated differently than functions – from the computational viewpoint, $<$ is simply a function. Just like $+$ is a function that takes two numbers and returns a number which is their sum, the relation $<$ is a function that takes two numbers and returns a boolean value: true or false.

Since $<$ can be naturally treated as function, the question of how to extend this to fuzzy becomes a particular case of a more general question of how to extend functions to fuzzy – and this extension is well known, it is described by Zadeh's extension principle. Let us recall how this principle is usually derived.

Zadeh's extension principle and how it is usually derived. Suppose that we have a function $y = f(x_1, \ldots, x_n)$ of n real-valued variables, and we have fuzzy information about the values $x_1, \ldots, x_n$, i.e.,, we know membership functions $\mu_1(x_1), \ldots, \mu_n(x_n)$ that describes our knowledge about the inputs $x_1, \ldots, x_n$. Based on this information, what do we know about $y = f(x_1, \ldots, x_n)$?

Intuitively, Y is a possible value of the variable y if there exists values $X_1, \ldots, X_n$ for which X_1 is a possible value of x_1 *and* … *and* X_n is a possible value of x_n *and* $Y = f(X_1, \ldots, X_n)$. We know the degrees

$\mu_i(X_i)$ to which each each real number X_i is a possible values of the input x_i. To combine these degrees into our degree of confidence in a composite and-statement, we can use an "and"-operation (t-norm), the simplest of which is $\min(a, b)$. Thus, for each tuple $(X_1, \ldots, x_n)$ for which $Y = f(X_1, \ldots, X_n)$, our degrees of confidence is the above and-statement is $\min(\mu_1(X_1), \ldots, \mu_n(X_n))$.

The existential quantifier "there exists" is, in effect, an "or": it means that either this property is true for one tuple, or for another tuple, etc. Thus, to find the degree to which the value Y is possible, we need to apply an "or"-operation (t-conorm) to the degrees of confidence of the corresponding and-statements. The simplest "or"-operation is $\max(a, b)$. Thus, we arrive at the following formula for the degree $\mu(Y)$ to which Y is a possible value of the variable y:

$$\mu(Y) = \max\{\min(\mu_1(X_1), \ldots, \mu_n(X_n)) : f(X_1, \ldots, X_n) = Y\}.$$

This formula – first proposed by L. Zadeh himself – is known as *Zadeh's extension principle.*

Let us apply Zadeh's extension principle to our problem: resulting formulas. In our case, we have a Boolean-valued function $f(x_1, x_2) = (x_1 < x_2)$ of $n = 2$ real-valued variables. When we compare an alternative a with fuzzy utility $\mu_a(u)$ with a crisp value u_0, Zadeh's extension principle takes the following form:

– for the value $y =$"true", the degree $\mu_+(a < u_0)$ that the statement $a < u_0$ is true is equal to

$$\mu_+(a < u_0) = \max(\mu_a(u) : u < u_0);$$

– for the value $y =$"false", the degree $\mu_-(a < u_0)$ that the statement $a < u_0$ is false is equal to

$$\mu_-(a < u_0) = \max(\mu_a(u) : u \geq u_0).$$

Let us analyze the resulting formulas. Intuitively, since in fuzzy logic negation is represented by the function $1 - a$ (in the sense that our degree of believe that A is false is estimated as 1 minus degree that A is true), we should expect that $\mu_+(a < u_0) + \mu_-(a < u_0) = 1$. Let us show, however, that this is not the case.

Indeed, let us consider a typical case when $\mu_a(u)$ is a fuzzy number, i.e., when for some value U:

– the function $\mu_a(u)$ increases to 1 when $u \leq U$, and
– this function decreases from 1 when $u \geq U$.

When $u_0 < U$, then the function $\mu_a(u)$ is increasing for all $u < u_0$ and thus, $\mu_+(a < u_0) = \mu_a(u_0)$. On the other hand, since $u_0 < U$ and for $u = U$, we have $\mu_a(U) = 1$, we get $\mu_-(a < u_0) = 1$. Thus,

$$\mu_+(a < u_0) + \mu_-(a < u_0) = 1 + \mu_a(u) \neq 1,$$

unless, of course, we consider absolutely impossible values u for which $\mu_a(u) = 0$.

Similarly, when $u_0 \geq U$, then the function $\mu_a(u)$ is decreasing for all $u > u_0$ and thus, $\mu_-(a < u_0) = \mu_a(u_0)$. On the other hand, since $u_0 \geq U$ and for $u = U$, we have $\mu_a(U) = 1$, we get $\mu_+(a < u_0) = 1$. Thus, in this case too, we have

$$\mu_+(a < u_0) + \mu_-(a < u_0) = 1 + \mu_a(u) \neq 1,$$

unless, of course, we consider absolutely impossible values u for which $\mu_a(u) = 0$.

So, we get intuitionistic fuzzy degrees. In the traditional fuzzy logic, the sum of degrees to which each statement is true and to which this same statement is false is always equal to 1. This means that when we compare alternatives, we get beyond the traditional fuzzy logic.

How can we describe where we are? This is not the only case when the degrees of confidence in a statement and in its negation doe not add up to 1. To describe such cases, K. Atanassov came up with an idea of *intuitionistic fuzzy logic* (see, e.g., [1]), in which, for each statement, we have *two* degrees:

- the degree to which this statement is true, and
- the degree to which this statement is false,

and these degrees do not necessarily add to 1. Our analysis this leads us to a conclusion that the result of comparing two alternatives is an intuitionistic fuzzy degree.

3 Discussion

What we got is somewhat different from intuitionistic fuzzy logic. There is a minor difference between what we observe when comparing two alternative and the traditional intuitionistic fuzzy logic is that:

- in the intuitionistic fuzzy logic, the sum of positive and negative degrees is always smaller than or equal to 1, while
- in our case, the sum is always greater than or equal to 1.

However, such (minor) generalization of intuitionistic fuzzy logic has been proposed in the past.

There is also a way to reconcile the results of comparing alternatives with the traditional intuitionstic fuzzy logic. Indeed, in general, Zadeh's extension principle, we compute the degree to which y is a *possible* value. In particular, $\mu_+(a < u_0)$ is the degree to which is is possible that $a < u_0$, and $\mu_-(a < u_0)$ is a degree to which it is possible that $a \geq u_0$. Instead, we can consider degrees $n_+(a < u_0)$ and $n_-(a < u_0)$ to which it is *necessary* that $a < u_0$ and that $a \geq u_0$ – defined, as usual, as 1 minus the degree to which the opposite statement is possible. Then, we get

$$n_+(a < u_0) = 1 - \mu_-(a < u_0)$$

and

$$n_-(< u_0) = 1 - \mu_+(a < u_0).$$

From the fact that $\mu_+(a < u_0) + \mu_-(a < u_0) \geq 1$, we can now conclude that

$$n_+(a < u_0) + n_-(a < u_0) = 2 - (\mu_+(a < u_0) + \mu_-(a < u_0)) \leq 1.$$

Thus, the degrees of necessity are consistent with the traditional intuionistic fuzzy logic.

We can still reconstruct the original membership function from the results of expert elicitation. We assume that the expert's preferences are described by a membership function $\mu_a(u_0)$. As we have mentioned, as a result of expert elicitation, we do not get this function, we get instead a more complex construct, in which for each possible value u_0, we get two degrees $\mu_+(a < u_0)$ and $\mu_-(a < u_0)$.

We should mention, however, that from this construct, we can uniquely reconstruct the original membership function. Indeed, as have shown:

- when $u_0 \le U$, then we have $\mu_+(a < u_0) = \mu_a(u_0)$ and $\mu_-(a < u_0) = 1$; and
- when when $u_0 \ge U$, then we have $\mu_-(a < u_0) = \mu_a(u_0)$ and $\mu_+(a < u_0) = 1$.

In both cases, we thus have

$$\mu_a(u_0) = \min(\mu_+(a < u_0), \mu_-(a < u_0)).$$

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