SUMMATION OF DIVERGENT INFINITE SERIES: HOW NATURAL ARE THE CURRENT TRICKS

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Abstract: Infinities are usually an interesting topic for students, especially when they lead to what seems like paradoxes, when we have two different seemingly correct answers to the same question. One of such cases is summation of divergent infinite sums: on the one hand, the sum is clearly infinite, on the other hand, reasonable ideas lead to a finite value for this same sum. A usual way to come up with a finite sum for a divergent infinite series is to find a 1-parametric family of series that includes the given series for a specific value $p = p_0$ of the corresponding parameter and for which the sum converges for some other values p. For the values p for which this sum converges, we find the expression s(p) for the resulting sum, and then we use the value $s(p_0)$ as the desired sum of the divergent infinite series. To what extent is the result reasonable depends on how reasonable is the corresponding generalizing family. In this paper, we show that from the physical viewpoint, the existing selection of the families is very natural: it is in perfect accordance with the natural symmetries.

Keywords: divergent infinite series, symmetries

Summation of divergent infinite series: an interesting topic. Infinities are mysterious. Not surprisingly, topics related to infinities are often exciting for students – especially when it turns out that what seemed simple and straightforward in the finite case is no longer simple and no longer straightforward.

One such case is the summation of infinite series. At first glance, this seems to be a straightforward topic:

- some series converge and have a finite sum, while
- some series diverge – e.g., if the resulting sum is infinite.

However, an interesting part is that often,

- while the usual methods lead to an infinite value of the corresponding sum,
- other techniques lead us to a finite value for the sum of the same series.

Let us start with an example of how we can get such a divergent infinite series with a finite sum.

Summation of divergent infinite series: first example. In many cases, we can get an explicit formula for the sum of an infinite series – by properly manipulating this series. For example, for an infinite geometric progression

$$s = 1 + p + p^2 + p^3 + \ldots + p^n + \ldots$$

we can multiply this sum by p, add 0 in front, and get

$$s * p = 0 + p + p^2 + p^3 + \ldots + p^n + \ldots$$

If we now subtract the new series from the original one term-by-term, all the terms in the right-hand side disappear except for the first terms 1, so we conclude that $s * (1 - p) = 1$ and thus, that

$$s = 1/(1 - p).$$

So, for the values p from –1 to 1, for which the sum of the geometric progression converges, we get the correct expression for this sum. For example, for p = 0.5, we get s = 1/(1 – 0.5) = 2, this we get

$$1 + 0.5 + 0.5^2 + 0.5^3 + \ldots + 0.5^n + \ldots = 2.$$

Interestingly, the above trick can be applied when the value p is outside the open interval (--1, 1). For example, for p = 2, when the above infinite series clearly diverges, we get

$$s = 1/(1 – 2) = --1,$$

thus we get a finite sum for the divergent infinite series:

$$1 + 2 + 2^2 + 2^3 + \ldots + 2^n + \ldots = --1.$$

Summation of divergent infinite series: general idea. The above idea shows how, in general, we can come up with a meaningful finite expression for the divergent infinite series:

- We start with a divergent infinite series for which we want to compute the sum

  $$s = a_0 + a_1 + a_2 + \ldots + a_n + \ldots$$

- We then find a 1-parametric family of infinite series that includes the desired series as a particular case, and which is convergent for some values of the corresponding parameter p:

  $$s(p) = a_0(p) + a_1(p) + a_2(p) + \ldots + a_n(p) + \ldots$$

- For the cases when the sum converges, we find the explicit expression for s(p), and then apply this expression to the value p_0 corresponding to the original series. The resulting value $s(p_0)$ is then returned as the sum of the original divergent infinite series.

First example reformulated in these general terms. Let us show, in detail, that the above derivation is a particular case of this general idea.

In this example, we want to compute the sum

$$s = 1 + 2 + 2^2 + 2^3 + \ldots + 2^n + \ldots$$

This sum is divergent, so we find a family of series that includes this sum as a particular case corresponding to $p_0 = 2$:

$$s(p) = 1 + p + p^2 + p^3 + \ldots + p^n + \ldots$$

This sum is convergent for some values of the parameter x: namely, for all the values p from –1 to 1. For these values, s(p) = 1/(1 – p). To find the desired value of the sum s, we thus substitute p_0 = 2 into this formula and get $s = s(p_0) = s(2) = 1/(1 – 2) = --1$.

Second example. Let us illustrate the above general idea on another known example of a divergent series: computing the sum of an infinite arithmetic progression

$$s = 1 + 2 + 3 + 4 + \ldots$$

To compute this sum, it turned out to be useful to utilize the following family:

$$s = 1^p + 2^p + 3^p + 4^p + \ldots$$

The original sum corresponds to $p_0 = 1$. The new series is convergent for all values $p < -1$, e.g., for $p = -2$.

To compute the value s, instead of multiplying the sum by x as in the first example, let us multiply it by $2*2^p$. After the multiplication 1^p becomes $2*2^p$, 2^p becomes $2*4^p$, 3^p becomes $2*6^p$, etc. Let us place 0s so that the term $2*2^p$ be at the same level as 2^p, etc. Then, we get:

$$2*2^p*s = 0 + 2*2^p + 0 + 2*4^p + \ldots$$

Subtracting the new expression from the original series term-by-term, we conclude that

$$c = 1^p - 2^p + 3^p - 4^p + \ldots,$$

where we denoted $c = s * (1 - 2 * 2^p)$.

Let us now shift the series by adding 0 in front:

$$c = - 0^p + 1^p - 2^p + 3^p - \ldots$$

By adding the above two expressions for c, we conclude that

$$2c = (1^p - 0^p) - (2^p - 1^p) + (3^p - 2^p) - (4^p - 3^p) + \ldots$$

Again, we shift the series by adding 0 in front:

$$2c = 0 + (1^p - 0^p) - (2^p - 1^p) + (3^p - 2^p) - \ldots$$

By adding the above two expressions for 2c, and taking into account that $1^p - 0^p = 1$, we conclude that

$$4c = 1 - (2^p - 2*1^p + 0^p) + (3^p - 2*2^p + 1^p) - (4^p - 2*3^p + 2^p) + \ldots$$

The advantage of this formula is that for value p not exceeding 1, the right-hand side is a convergent sum. In particular, for $p = p_0 = 1$, each term $(n + 1)^p - 2*n^p + (n - 1)^p$ in this sum is equal to

$$(n + 1) - 2*n + (n - 1) = 0.$$

Thus, in the expression for 4c, the only non-zero term is the first 1, so $4c = 1$ and thus $c = 0.25$. By definition, $c = s * (1 - 2 * 2^p)$, i.e., for $p = 1$, we get $c = -3s$. Thus, we conclude that:

$$s = 1 + 2 + 3 + \ldots + n + \ldots = -1/12.$$

How natural is all this? A reasonable question is: OK, we used some tricks, and we got some reasonable results. But how natural are these tricks? Maybe if we used different tricks, we would have gotten different results?

In other words, how natural are the families that we chose, families that include the original series as a particular case? In general, we want to extend the family based on a single example. Of course, there are many different families that we could choose. How natural is the selection of the families p^n and n^p that we used in the above two examples?

Let us go back to the physical meaning of the series. From the purely mathematical viewpoint, it is difficult to see which families are natural and which are not natural: everything is purely mathematical and thus, seems to be not very natural.

So, to decide which families are natural and which families are not natural, let us take into account that the series are not just an abstract mathematical concept, they are actively used in describing the real world. Typically, infinite series appear when we measure the value a(t) of some physical quantity a at some sequential moments of time

$$t_0,\ t_1 = t_0 + h,\ t_2 = t_1 + h = t_0 + 2h,\ \ldots,\ t_n = t_{n-1} + h = t_0 + n * h,\ \ldots$$

so that $a_n = a(t_n)$.

So, to find out which series are natural, we need to analyze which function a(t) are natural.

Natural symmetries. To analyze which physical dependencies are natural, let us take into account that the numerical value of a physical quantity depends on the selection of the measuring unit. If instead of the original measuring unit we use another unit which is C times smaller, then all the numerical values get multiplied by this constant C.

For example, if instead of meters we consider centimeter, then all the numerical values get multiplied by 100: e.g., 2 meters becomes 200 centimeters.

From this viewpoint, the function a(t) and the function C * a(t) represent the exact same dependence of the quantity a on time t, but expressed in different measuring units.

For measuring time, we can also select different units. For time, we also have an additional freedom – in addition to selecting a different measuring unit, we can also select a different starting point. For example, during the French revolution, in the revolutionary calendar, the year of the revolution (1789 in the usual calendar) was officially designated as Year 1. In general, if as a new starting point, we select a starting point which is T moments earlier than the previous one, then all numerical values of time are increase by this amount T: $t \rightarrow t + T$.

Natural functions a(t). Physical processes do not change if we simply change measuring units or starting point for measurements. From this viewpoint, it is reasonable to consider a function a(t) natural if the corresponding physical process does not depend on the selection of a measuring unit for time or on the selection of the starting point.

Let us consider these two options one by one.

Case when the relation a(t) **does not depend on the starting point for measuring time.** Let us first consider the case when the relation a(t) does not depend on the starting point for measuring time.

If we change the starting point, the value t is replaced by t + T, and, correspondingly, the function a(t) get replaced by the new function a(t + T). We want to make sure that both the original function a(t) and the new function a(t + T) represent the same physical process. As we have mentioned earlier, this means that the functions a(t + T) and a(t) differ by a multiplicative constant – representing a change in the unit for measuring a: a(t + T) = C * a(t), for some constant C depending on T.

In particular, for T = h, this means that $a_{i+1} = a(t_{i+1}) = a(t_i + h) = C * a(t_i) = C * a_i$. So:

$$a_1 = C * a_0,\ a_2 = C * a_1 = C^2 * a_0, \text{ and, in general, } a_n = C^n * a_0.$$

Thus, we get – modulo a multiplicative constant a_0 – a geometric progression that was used in our first example.

Case when the relation $a(t)$ **does not depend on the selection of a measuring unit for measuring time.** Let us now consider the case when the relation $a(t)$ does not depend on the selection of a measuring unit for measuring time.

Changing such a unit leads to changing t to $c * t$, and $a(t)$ to $a(c * t)$. The fact that these two functions should describe the same physical process means that we should have $a(c * t) = C * a(t)$, for some constant C depending on c, i.e., that we should have

$$a(c * t) = C(c) * a(t).$$

It makes sense to require that the dependence $a(t)$ is described by a measurable (= integrable) function – otherwise, we will not be able to integrate it, while what we usually observe is not the instantaneous value but rather an average (weighted integral) over some time interval including the moment t. For measurable functions, it is known that all solutions of the above functional equation have the form $a(t) = c * t^p$ for some constants c and p; see, e.g., [1].

By changing the starting point for measuring time, we can always take $t_0 = 0$; then, $t_n = n * h$. Thus, we have:

$$a_n = a(t_n) = a(n * h) = c * (n * h)^p = n^p * (c * h^p).$$

So, modulo a multiplicative constant, we get the dependence n^p that was used in the second example.

Conclusion. Our conclusion is that while generalizations used to compute the above two sums of divergent series sound somewhat arbitrary, in reality, these generalizations are very natural – they follow directly from the requirement that the corresponding physical relation not change if we change

- either the starting point for measuring time,
- or the measuring unit for time.

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References

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