

# Why Triangular and Trapezoid Membership Functions Are Efficient in Design Applications

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## Abstract

In many design problems, it is important to take into account expert knowledge. Experts often describe their knowledge by using imprecise (“fuzzy”) natural-language words like “small”. To describe this imprecise knowledge in computer-understandable terms, Zadeh came up with special *fuzzy* methodology – techniques that have been successful in many applications. This methodology starts with eliciting, from the expert, a membership function corresponding to each imprecise term – a function that assigns, to each possible value of the corresponding quantity, a degree to which this value satisfies the relevant property (e.g., a degree to which, in the expert’s opinion, this value is small). In principle, we can have complex membership functions. However, somewhat surprisingly, in many applications, the simplest membership functions – of triangular or trapezoid shape – turned out to be most efficient. There exist some explanations for this surprising empirical phenomenon, but these explanations only work when we use the simplest possible “and”-operation – minimum. In this paper, we provide a new, more general explanation which is applicable for all possible “and”-operations.

## 1 Formulation of the Problem

**Expert knowledge is important in design.** In engineering design, it is often very important to take into account the human expertise.

**Taking expert knowledge into account is often not easy.** Taking into account expert’s knowledge is not always easy, since experts often formulate their knowledge not in precise computer-understandable terms, but by using imprecise (fuzzy) words from natural language such as “small”, “approximately”, “close”, etc.

**Fuzzy techniques – a way to take into account imprecise expert knowledge.** To take imprecise expert knowledge into account, Lotfi Zadeh came up with special *fuzzy techniques*; see, e.g., [1, 2, 4, 5, 6, 7].

**First stage of fuzzy methodology: eliciting membership functions.** In the fuzzy approach, first, we describe the meaning of the corresponding natural-language words in precise numerical terms. For this purpose, for each such word (e.g., for the word “small”):

- we provide the expert with several possible values  $x$  of the corresponding quantity,
- for each of these values, we ask the expert to mark a point, on a scale from 0 to 1, to what extent the given value satisfies this property (e.g., to what extent this value is small).

The resulting points  $\mu_P(x) \in [0, 1]$  corresponding to different values  $x$  form what is called a *membership function* or a *fuzzy set* describing the corresponding natural-language property  $P$ .

**Second stage of fuzzy methodology: combining membership degrees.**

An additional problem stems from the fact that many expert rules have several conditions: e.g., “if the road will have heavy traffic and the region experiences drastic changes from freezing to thawing, then the road pavement must be made reasonably resilient to such changes.” The condition to this rule contains two imprecise natural-language words: “heavy” and “drastic”. In the ideal world,

- in addition to asking the experts for all possible values of  $\mu_{\text{heavy}}(t)$  and  $\mu_{\text{drastic}}(c)$ ,
- we should also ask, for all possible combinations of traffic  $t$  and temperature change  $c$ , to what extent the above condition is satisfied for the corresponding pair  $(t, c)$ .

However, in practice, already asking the expert about all possible values of one quantity takes a long time. So, asking the expert about all possible pairs is not realistically possible.

Since we cannot elicit, from the expert, the degree to which each pair satisfies the corresponding condition, we must therefore estimate this degree based on what we know, i.e., on the degrees to which the traffic is heavy and to which the temperature change is drastic. In general, we know the degrees  $a$  and  $b$  to which imprecise statements  $A$  and  $B$  are true, and based on these degrees, we need to estimate the degree to which the “and”-combination  $A \& B$  is true. Let us denote the corresponding estimate by  $f_{\&}(a, b)$ .

The algorithm that computes these estimates based on the two given degrees is known as an “*and*”-operation or a *t-norm*. The t-norm has to satisfy several reasonable properties. For example, since  $A \& B$  means the same as  $B \& A$ , we expect that the estimates for these two “and”-combinations should be the same, i.e., that  $f_{\&}(a, b) = f_{\&}(b, a)$  for all  $a$  and  $b$ . Similarly, since  $A \& (B \& C)$  and  $(A \& B) \& C$  mean the same, we expect that the resulting estimates are equal, i.e., that  $f_{\&}(a, f_{\&}(b, c)) = f_{\&}(f_{\&}(a, b), c)$  for all  $a, b$ , and  $c$ .

All t-norms that satisfy all these properties are known. The most widely used “and”-operations are  $f_{\&}(a, b) = \min(a, b)$  and  $f_{\&}(a, b) = a \cdot b$ , but many other operations are also used.

**Triangular and trapezoid membership functions are usually very efficient.** In principle, if we ask an expert, we can get different shapes of membership functions. In the beginning, practitioners tried to describe these membership functions as accurately as possible. However, it soon turned out that in most applications, it is sufficient to consider simple membership functions whose graphs have triangular or trapezoid shape.

To be more precise, triangular membership functions are the following functions which are different from 0 on an interval  $[\underline{x}, \bar{x}]$  with midpoint  $\tilde{x}$ :

- they linearly increase from 0 to 1 when  $x$  is smaller than the midpoint  $\tilde{x}$  and then
- they linearly decrease from 1 to 0 when  $x$  is larger than the midpoint  $\tilde{x}$ .

In precise terms:

- $\mu(x) = \frac{x - \underline{x}}{\tilde{x} - \underline{x}}$  when  $\underline{x} \leq x \leq \tilde{x}$ ;
- $\mu(x) = \frac{\bar{x} - x}{\bar{x} - \tilde{x}}$  when  $\tilde{x} \leq x \leq \bar{x}$ , and
- $\mu(x) = 0$  when  $x < \underline{x}$  or when  $x > \bar{x}$ .

Trapezoid membership functions also have an interval in the middle when the function is identically equal to 1. To be more precise, we select four values  $\underline{x} < \underline{t} < \bar{t} < \bar{x}$  and then we take:

- $\mu(x) = 0$  for  $x \leq \underline{x}$ ;
- $\mu(x) = \frac{x - \underline{x}}{\underline{t} - \underline{x}}$  when  $\underline{x} \leq x \leq \underline{t}$ ;
- $\mu(x) = 1$  when  $\underline{t} \leq x \leq \bar{t}$ ,
- $\mu(x) = \frac{\bar{x} - x}{\bar{x} - \bar{t}}$  when  $\bar{t} \leq x \leq \bar{x}$ , and
- $\mu(x) = 0$  for  $x \geq \bar{x}$ .

**Open problem: why are these membership functions efficient?** While empirical evidence shows that triangular and trapezoid membership functions are efficient in many engineering applications, there is still no convincing general explanation for this empirical efficiency.

A partial explanation was provided in [3], but this explanation is only valid when we use a minimum t-norm. In this paper, we show that a similar explanation can be made general by extending it to general “and”-operations.

## 2 Main Idea Behind Known Partial Explanation: A Brief Reminder

To explain this main idea, let us recall, in some detail, how membership functions are usually elicited.

**Elicitation of membership functions: first step.** Usually, for each property, there is a threshold  $\underline{x}$  below which, according to the expert, the corresponding property is definitely not satisfied. For example, experts may have different opinions on what constitutes warm, but for every expert, there is a temperature below which it is clearly not warm. For some people from the North, this threshold may be 14 C, for people from the tropics, even 19 C may be chilly, so for them it will be around 22 C, but there is such threshold for every expert.

In precise terms, this means that  $\mu(x) = 0$  for all  $x \leq \underline{x}$ .

**Elicitation of membership functions: second step.** Similarly, there exists a threshold  $\bar{x}$  above which the corresponding property is definitely not satisfied. This means that  $\mu(x) = 0$  for all  $x \geq \bar{x}$ .

**Elicitation of membership functions: third step.** There also usually exist values for which the corresponding property is definitely satisfied. For example, for “warm”, most people will agree that 25 C is warm.

Sometimes, there is a single value  $\tilde{x}$  for which the original property is definitely satisfied. Sometimes, there is a whole interval of values  $[\underline{t}, \bar{t}]$  for all of which the given property is definitely satisfied. For all such values  $x$ , we have  $\mu(x) = 1$ .

**Elicitation of membership functions: fourth step.** For the intervals  $[\underline{x}, \tilde{x}]$  and  $[\tilde{x}, \bar{x}]$  (or, alternatively,  $[\underline{x}, \underline{t}]$  and  $[\bar{t}, \bar{x}]$ ), the membership degrees change from 0 to 1 and then from 1 to 0. These are the values that we need to elicit from the expert.

For each of the intervals, to elicit these values, we provide the expert with several values  $x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n$  from the corresponding interval – where  $x_0$  and  $x_n$  are the interval’s endpoints – and elicit the corresponding membership degrees  $\mu_i \stackrel{\text{def}}{=} \mu(x_i)$  for  $i = 1, \dots, x_n$  (the values  $\mu_0$  and  $\mu_n$  corresponding to endpoints are known). Usually, the values  $x_i$  are equally spaced:  $x_1 - x_0 = x_2 - x_1 = \dots$ , so that  $\mu_i = \mu_0 + i \cdot h$ , where we denoted  $h \stackrel{\text{def}}{=} x_1 - x_0$ .

**Main idea behind the known (partial) explanation of the effectiveness of triangular and trapezoid membership functions.** When the value  $x'$  is close to the value  $x$  ( $x' \approx x$ ), we do not expect that the expert’s degree of confidence  $\mu(x')$  that  $x'$  satisfies the given property to be much different from the expert’s degree of confidence  $\mu(x)$  that the original value  $x$  satisfies this property: we should have  $\mu(x) \approx \mu(x')$ . For example:

- if  $x$  is reasonably small, and
- $x'$  is close to  $x$ ,

- then it is reasonable to conclude that  $x'$  should also be reasonably small – with almost the same degree of smallness as  $x$ .

In other words, if  $x$  and  $x'$  are close, then the values  $\mu(x)$  and  $\mu(x')$  should also be close. For large  $n$ , the differences between  $x_{i-1}$  and  $x_i$  are small, so  $x_{i-1}$  is close to  $x_i$ . Thus, we conclude that for every  $i$ , the values  $\mu_{i-1}$  and  $\mu_i$  should be close.

The more this property is satisfied, the more it is reasonable that the values  $\mu_i$  adequately describe the expert's knowledge. It thus makes sense to select the values  $\mu_i$  that satisfy the above property to the largest possible degree. In precise terms:

- for each possible sequence of the values  $\mu_i$ , we can find the degree to which the above property is satisfied, and then
- as the most adequate description of the expert's knowledge, we select the values  $\mu_i$  for which this degree is the largest possible.

How can we describe this degree? What does it mean that two values  $\mu_{i-1}$  and  $\mu_i$  are close? Intuitively, it means that the absolute value of their difference  $|\mu_i - \mu_{i-1}|$  is small. Let  $s(v)$  denote the membership function corresponding to “small”. Then, for each  $i$ , the degree to which  $\mu_i$  and  $\mu_{i-1}$  are close is equal to  $s(|\mu_i - \mu_{i-1}|)$ , and the degree to which this closeness condition is satisfied for  $i = 1$  and for  $i = 2$ , etc., is equal to

$$f_{\&}(s(|\mu_1 - \mu_0|), s(|\mu_2 - \mu_1|), \dots, s(|\mu_n - \mu_{n-1}|)), \quad (1)$$

for an appropriate “and”-operation  $f_{\&}(a, b)$ .

We must find the values  $\mu_1, \dots, \mu_{n-1}$  for which the expression (1) attains the largest possible value.

### 3 Analysis of the Problem and the Resulting Justification

**It is reasonable to expect that this optimization problem has a unique solution.** In engineering problems, in principle, we may have optimality criteria that allow many different solutions. For the same problem, we have several solutions which are equally good according to the selected criterion. However, from the practical viewpoint, this means that the corresponding criterion is not final.

For example, if we have several plane designs with similar energy consumption and similar manufacturing and exploitation costs, this means that we can select, among these designs, the one with the smallest negative effect on the environment – and thus narrow down the set of all optimal designs. Eventually, we will end up with a final optimality criterion for which exactly one alternative is optimal.

From this viewpoint, it is reasonable to require that our criterion (1) is, in this sense, final – i.e., that it leads to the unique selection of the corresponding membership degrees  $\mu_i$ .

Let us describe this requirement in precise terms.

**Definition 1.**

- By a membership function, we will (as usual) mean a mapping  $s(v)$  from real numbers to the interval  $[0, 1]$ .
- By an  $n$ -aggregation operation, we mean a function  $f(a_1, \dots, a_n)$  of  $n$  variables  $a_i \in [0, 1]$  with values from  $[0, 1]$  which is symmetric, i.e., for which:

$$f(a_1, \dots, a_n) = f(a_{\pi(1)}, \dots, a_{\pi(n)})$$

for all  $a_i$  and for any permutation  $\pi$ .

- We say that a pair  $\langle f_{\&}, s \rangle$ , where  $f_{\&}$  is an  $n$ -aggregation operation and  $s$  is a membership function, is final for increasing sequences if there exists exactly one sequence of values  $\mu_0 = 0 < \mu_1 < \mu_2 < \dots < \mu_n = 1$  for which the expression (1) attains its largest possible value. The corresponding sequence  $\mu_i$  will be called the optimal increasing sequence.

**Proposition 1.** For every pair  $\langle f_{\&}, s \rangle$  which is final for increasing sequences, the optimal increasing sequence has the form  $\mu_i = \frac{i}{n}$ .

*Comments.*

- For reader’s convenience, the proof is given in the special (last) Proofs section.
- According to this result, for the optimal increasing section of the membership function, we have  $\mu(x_i) = \mu(x_0 + i \cdot h) = \mu_i = \frac{i}{n}$ . Let us describe  $\mu(x_i)$  in terms of  $x_i$ . From  $x_i = x_0 + i \cdot h$ , we conclude that  $i = \frac{x_i - x_0}{h}$ , thus  $\mu(x_i) = \frac{x_i - x_0}{n \cdot h}$ , i.e.,  $\mu(x_i) = a \cdot x_i + b$ , where  $a \stackrel{\text{def}}{=} \frac{1}{n \cdot h}$  and  $b \stackrel{\text{def}}{=} -\frac{x_0}{n \cdot h}$ . So, the optimal membership function  $\mu(x)$  on the interval  $[\underline{x}, \tilde{x}]$  is a linear function that takes the value 0 for  $x = \underline{x}$  and the value 1 for  $x = \tilde{x}$ , thus,  $\mu(x) = \frac{x - \underline{x}}{\tilde{x} - \underline{x}}$ . This is exactly the increasing linear segment that we have been trying to explain.
- Proposition 1 does not require that the  $n$ -aggregation operation is minimum – it can be any t-norm. Moreover, we do not even require that this operation be a t-norm: we do not require its associativity or monotonicity. So, our result is even more general than what we wanted.
- A similar result holds for optimal decreasing sequences:

**Definition 2.** We say that a pair  $\langle f_{\&}, s \rangle$ , where  $f_{\&}$  is an  $n$ -aggregation operation and  $s$  is a membership function, is final for decreasing sequences if there exists exactly one sequence of values  $\mu_0 = 1 > \mu_1 > \mu_2 > \dots > \mu_n = 0$  for which the expression (1) attains its largest possible value. The corresponding sequence  $\mu_i$  will be called the optimal decreasing sequence.

**Proposition 2.** For every pair  $\langle f_{\&}, s \rangle$  which is final for decreasing sequences, the optimal decreasing sequence has the form  $\mu_i = 1 - \frac{i}{n}$ .

*Comment.* According to this result, for the optimal decreasing section of the membership function, we have  $\mu(x_i) = \mu(x_0 + i \cdot h) = \mu_i = 1 - \frac{i}{n}$ . Let us describe  $\mu(x_i)$  in terms of  $x_i$ . From  $x_i = x_0 + i \cdot h$ , we have  $i = \frac{x_i - x_0}{h}$ , thus  $\mu(x_i) = 1 - \frac{x_i - x_0}{n \cdot h}$ , i.e.,  $\mu(x_i) = a \cdot x_i + b$ , where  $a \stackrel{\text{def}}{=} -\frac{1}{n \cdot h}$  and  $b \stackrel{\text{def}}{=} 1 + \frac{x_0}{n \cdot h}$ .

So, the optimal decreasing membership function  $\mu(x)$  on the interval  $[\tilde{x}, \bar{x}]$  is a linear function that takes the value 1 for  $x = \tilde{x}$  and the value 0 for  $x = \bar{x}$ , thus,  $\mu(x) = \frac{\bar{x} - x}{\bar{x} - \tilde{x}}$ . This is exactly the decreasing linear segment that we have been trying to explain.

## 4 Proof of Propositions 1 and 2

It is sufficient to prove Proposition 1, the proof of Proposition 2 is similar.

Let us prove, by contradiction, that for the optimal increasing sequence  $\mu_i$ , we have  $\mu_1 - \mu_0 = \mu_2 - \mu_1 = \dots$ , i.e., that all the differences  $\Delta_i \stackrel{\text{def}}{=} \mu_i - \mu_{i-1}$  are equal to each other.

Indeed, assume that some of these differences are different, i.e.,  $\Delta_{i'} \neq \Delta_{i''}$  for some  $i'$  and  $i''$ . Without losing generality, we can assume that  $i' < i''$ . In this case, we can form the following auxiliary sequence  $\mu'_i$ :

- for  $i < i'$ , we take  $\mu'_i = \mu_i$ ;
- for  $i = i'$ , we take  $\mu'_{i'} = \mu_{i'-1} + \Delta_{i''}$ ;
- when  $i' < i < i''$ , we take  $\mu'_i = \mu_i + (\Delta_{i''} - \Delta_{i'})$ ;
- finally, for  $i \geq i''$ , we again take  $\mu'_i = \mu_i$ .

This sequence is different from the original sequence  $\mu_i$ , since  $\Delta_{i'} \neq \Delta_{i''}$  and thus,  $\mu'_{i'} = \mu_{i'-1} + \Delta_{i''} \neq \mu_{i'-1} + \Delta_{i'} = \mu_{i'}$ .

On the other hand, the differences  $\Delta'_i \stackrel{\text{def}}{=} \mu'_i - \mu'_{i-1}$  corresponding to the new sequence has the following form:

- for  $i < i'$ , we have  $\Delta'_i = \Delta_i$ ,
- for  $i = i'$ , we have  $\Delta'_{i'} = \Delta_{i''}$ ;

- when  $i' < i < i''$ , we get  $\Delta'_i = \Delta_i$ ;
- for  $i = i''$ , we get  $\Delta'_{i''} = \Delta'_{i'}$ ; and
- for  $i > i''$ , we again get  $\Delta'_i = \Delta_i$ .

Thus, the new sequence of differences  $\Delta'_i$  is obtained from the original sequence of differences  $\Delta_i$  by an appropriate permutation  $\pi$ : namely, by a permutation that swaps the indices  $i'$  and  $i''$  and leaves all the other indices unchanged. Thus, the values  $s(\Delta'_i) = s(|\mu'_{i'} - \mu'_{i'-1}|)$  are also obtained from the values  $s(\Delta_i) = s(|\mu_i - \mu_{i-1}|)$  by a similar permutation.

Since we assumed that the  $n$ -aggregation operation is symmetric, i.e., that the result of applying this operation does not change if we simply permute the inputs, we thus conclude that

$$\begin{aligned} f_{\&}(s(|\mu'_1 - \mu'_0|), s(|\mu'_2 - \mu'_1|), \dots, s(|\mu'_n - \mu'_{n-1}|))) = \\ f_{\&}(s(|\mu_1 - \mu_0|), s(|\mu_2 - \mu_1|), \dots, s(|\mu_n - \mu_{n-1}|))), \end{aligned} \quad (2)$$

i.e., that the expression (1) attains the same value for both sequences  $\mu_i$  and  $\mu'_i$ .

Since we assumed that the sequence  $\mu_i$  is optimal, this means that for this sequence, the value of the expression (1) is the largest possible. Thus, the above equality (2) shows that the value of the expression (1) for the new sequence  $\mu'_i$  is also optimal. So, we have two different sequences  $\mu_i$  and  $\mu'_i$  on which the expression (1) attains its maximum – which contradicts to our assumption that the pair  $\langle f_{\&}, s \rangle$  is final for increasing sequences.

This contradiction shows that the differences  $\Delta_i$  cannot be different, so they are all equal:  $\Delta_1 = \Delta_2 = \dots = \Delta_n$ , i.e.,  $\mu_1 - \mu_0 = \mu_2 - \mu_1 = \dots = \Delta_1$ . Thus:

- From  $\mu_0 = 0$ , we conclude that  $\mu_1 = \Delta_1$ .
- From  $\mu_2 - \mu_1 = \Delta_1$ , we conclude that  $\mu_2 = \mu_1 + \Delta_1 = 2\Delta_1$ .
- By induction, once we have shown that  $\mu_i = i \cdot \Delta_1$ , we can conclude that  $\mu_{i+1} = \mu_i + \Delta_1 = i \cdot \Delta_1 + \Delta_1$  and thus, that  $\mu_{i+1} = (i + 1) \cdot \Delta_1$ .
- So, we conclude that  $\mu_i = i \cdot \Delta_1$  for all  $i$ .
- In particular, for  $i = n$ , we conclude that  $\mu_n = n \cdot \Delta_1$ , hence  $\Delta_1 = \frac{1}{n}$  and thus,  $\mu_i = i \cdot \Delta_1 = \frac{i}{n}$ .

The proposition is proven.

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