Decision Making Under Interval Uncertainty: Beyond Hurwicz Pessimism-Optimism Criterion

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Abstract In many practical situations, we do not know the exact value of the quantities characterizing the consequences of different possible actions. Instead, we often only known lower and upper bounds on these values, i.e., we only know intervals containing these values. To make decisions under such interval uncertainty, the Nobelist Leo Hurwicz proposed his optimism-pessimism criterion. It is known, however, that this criterion is not perfect: there are examples of actions which this criterion considers to be equivalent but which for which common sense indicates that one of them is preferable. These examples mean that Hurwicz criterion must be extended, to enable us to select between alternatives that this criterion classifies as equivalent. In this paper, we provide a full description of all such extensions.

1 Formulation of the Problem

Decision making in economics: ideal case. In the ideal case, when we know the exact consequence of each action, a natural idea is to select an action that will lead to the largest profit.

Need for decision making under interval uncertainty. In real life, we rarely know the exact consequence of each action. In many cases, all we know are the lower and upper bound on the quantities describing such consequences, i.e., all we know is an interval $[\underline{a},\overline{a}]$ that contains the actual (unknown) value a.

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How can make a decision under such interval uncertainty? If we have several alternatives a for each of which we only have an interval estimate $[\underline{u}(a),\overline{u}(a)]$, which alternative should we select?

Hurwicz optimism-pessimism criterion. The problem of decision making under interval uncertainty was first handled by a Nobelist Leo Hurwicz; see, e.g., [2, 4, 5].

Hurwicz's main idea was as follows. We know how to make decisions when for each alternative, we know the exact value of the resulting profit. So, to help decision makers make decisions under interval uncertainty, Hurwicz proposed to assign, to each interval $\mathbf{a} = [\underline{a},\overline{a}]$, an equivalent value $u_H(\mathbf{a})$, and then select an alternative with the largest equivalent value.

Of course, for the case when we know the exact consequence a, i.e., when the interval is degenerate $[a,a]$, the equivalent value should be just a: $u_H([a,a]) = a$.

There are several natural requirements on the function $u_H(\mathbf{a})$. The first is that since all the values a from the interval $[\underline{a},\overline{a}]$ are larger than (thus better than) or equal to the lower endpoint $\underline{a}$, the equivalent value must also be larger than or equal to $\underline{a}$. Similarly, since all the values a from the interval $[\underline{a},\overline{a}]$ are smaller than (thus worse than) or equal to the upper endpoint $\overline{a}$, the equivalent value must also be smaller than or equal to $\overline{a}$:

$$\underline{a} \le u_H([\underline{a},\overline{a}]) \le \overline{a}.$$

The second natural requirement on this function is that the equivalent value should not change if we change a monetary unit: what was better when we count in dollars should also be better when we use Vietnamese Dongs instead. A change from the original monetary unit to a new unit which is k times smaller means that all the numerical values are multiplied by k. Thus, if we have $u_H(\underline{a},\overline{a}) = a_0$, then, for all $k > 0$, we should have

$$u_H([k\cdot\underline{a},k\cdot\overline{a}]) = k\cdot a_0.$$

The third natural requirement is related to the fact that if have two separate independent situations with interval uncertainty, with possible profits $[\underline{a},\overline{a}]$ and $[\underline{b},\overline{b}]$, then we can do two different things:

- first, we can take into account that the overall profit of these two situations can take any value from $\underline{a}+\underline{b}$ to $\overline{a}+\overline{b}$, and compute the equivalent value of the corresponding interval
$$\mathbf{a}+\mathbf{b} \stackrel{\text{def}}{=} [\underline{a}+\underline{b},\overline{a}+\overline{b}],$$
- second, we can first find equivalent values of each of the intervals and then add them up.

It is reasonable to require that the resulting value should be the same in both cases, i.e., that we should have

$$u_H([\underline{a}+\underline{b},\overline{a}+\overline{b}]) = u_H([\underline{a},\overline{a}]) + h_H([\underline{b},\overline{b}]).$$

This property is known as *additivity*.

These three requirements allow us to find an explicit formula for the equivalent value $h_H(\mathbf{a})$. Namely, let us denote $\alpha_H \stackrel{\text{def}}{=} u_H([0,1])$. Due to the first natural requirement, the value α_H is itself between 0 and 1: $0 \le \alpha_H \le 1$. Now, due to scale-invariance, for every value $a > 0$, we have $u_H([0,a]) = \alpha_H \cdot a$. For $a = 0$, this is also true, since in this case, we have $u_H([0,0]) = 0$. In particular, for every two values $\underline{a} \le \overline{a}$, we have $u_H([0,\overline{a}-\underline{a}]) = \alpha_H \cdot (\overline{a}-\underline{a})$.

Now, we also have $u_H([\underline{a},\underline{a}]) = \underline{a}$. Thus, by additivity, we get

$$u_H([\underline{a},\overline{a}]) = (\overline{a}-\underline{a}) \cdot \alpha_H + \underline{a},$$

i.e., equivalently, that

$$u_H([\underline{a},\overline{a}]) = \alpha_H \cdot \overline{a} + (1-\alpha_H) \cdot \underline{a}.$$

This is the formula for which Leo Hurwicz got his Nobel prize. The meaning of this formula is straightforward:

- When $\alpha_H = 1$, this means that the equivalent value is equal to the largest possible value $\overline{a}$. So, when making a decision, the person only takes into account the best possible scenario and ignores all other possibilities. In real life, such a person is known as an *optimist*.
- When $\alpha_H = 0$, this means that the equivalent value is equal to the smallest possible value $\underline{a}$. So, when making a decision, the person only takes into account the worst possible scenario and ignores all other possibilities. In real life, such a person is known as an *pessimist*.
- When $0 < \alpha_H < 1$, this means that a person takes into account both good and bad possibilities.

Because of this interpretation, the coefficient α_H is called *optimism-pessimism coefficient*, and the whole procedure is known as *optimism-pessimism criterion.*

Need to go beyond Hurwicz criterion. While Hurwicz criterion is reasonable, it leaves several options equivalent which should not be equivalent. For example, if $\alpha_H = 0.5$, then, according to Hurwicz criterion, the interval $[-1,1]$ should be equivalent to 0. However, in reality:

- A risk-averse decision maker will definitely prefer status quo (0) to a situation $[-1,1]$ in which he/she can lose.
- Similarly, a risk-prone decision maker would probably prefer an exciting gambling-type option $[-1,1]$ in which he/she can gain.

To take this into account, we need to go beyond assigning a numerical value to each interval. We need, instead, to describe possible orders on the class of all intervals.

This is what we do in this paper.

2 Analysis of the Problem, Definitions, and the Main Result

For every two alternatives a and b, we want to provide the decision maker with one of the following three recommendations:

- select the first alternative; we will denote this recommendation by $b < a$;
- select the second alternative; we will denote this recommendation by $a < b$; or
- treat these two alternatives as equivalent ones; we will denote this recommendation by $a \sim b$.

Our recommendations should be consistent: e.g.,

- if we recommend that b is preferable to a and that c is preferable to b,
- then we should also recommend that c is preferable to a.

Such consistency can be described by the following definition:

Definition 1. *For every set A, by a* linear pre-order*, we mean a pair of relations $(<, \sim)$ for which the following properties are satisfied:*

- *for every a and b, exactly one of the three possibilities must be satisfied: $a < b$, or $b < a$, or $a \sim b$;*
- *for all a, we have $a \sim a$;*
- *for all a and b, if $a \sim b$, then $b \sim a$;*
- *for all a, b, and c, if $a \sim b$ and $b \sim c$, then $a \sim c$;*
- *for all a, b, and c, if $a < b$ and $b < c$, then $a < c$;*
- *for all a, b, and c, if $a < b$ and $b \sim c$, then $a \sim c$; and*
- *for all a, b, and c, if $a \sim b$ and $b < c$, then $a < c$.*

Discussion.

- To fully describe a linear pre-order, it is sufficient to describe when $a < b$: indeed, by definition, once we know the relation $<$, we can uniquely reconstruct $a \sim b$ since
$$a \sim b \Leftrightarrow (a \not< b \,\&\, b \not< a).$$
- We want to describe all possible linear pre-orders on the set of all possible intervals. Of course, when the intervals are degenerate – i.e., are, in effect, exact real numbers – this pre-order must coincide with the usual order on the set of real numbers. Also, similarly to the Hurwicz case, an interval $[\underline{a}, \overline{a}]$ cannot be worse than $\underline{a}$ and cannot be better than $\overline{a}$. Thus, we arrive at the following definition.

Definition 2. *A linear pre-order on the set of all possible intervals $\mathbf{a} = [\underline{a}, \overline{a}]$ is called* natural *if the following two properties are satisfied:*

- *for every two numbers a and b, we have*
$$[a, a] < [b, b] \Leftrightarrow a < b;$$
- *for every $\underline{a} \le \overline{a}$, we have $[\underline{a}, \overline{a}] \not< [\underline{a}, \underline{a}]$ and $[\overline{a}, \overline{a}] \not< [\underline{a}, \overline{a}]$.*

Discussion. It is reasonable to require that our linear pre-order does not change if we change a monetary unit.

Definition 3. *A linear pre-order on the set of all possible intervals is called* scale-invariant *if for every two intervals* $\mathbf{a} = [\underline{a}, \overline{a}]$ *and* $\mathbf{b} = [\underline{b}, \overline{b}]$ *and for all real numbers* $k > 0$*, the following two implications hold:*

- *if* $[\underline{a}, \overline{a}] < [\underline{b}, \overline{b}]$*, then* $[k \cdot \underline{a}, k \cdot \overline{a}] < [k \cdot \underline{b}, k \cdot \overline{b}]$;
- *if* $[\underline{a}, \overline{a}] \sim [\underline{b}, \overline{b}]$*, then* $[k \cdot \underline{a}, k \cdot \overline{a}] \sim [k \cdot \underline{b} + \ell, k \cdot \overline{b} + \ell]$.

Discussion. Our next property is additivity.

Definition 4. *A linear pre-order on the set of all possible intervals is called* additive *if for every three intervals* $\mathbf{a}$*,* $\mathbf{b}$*, and* $\mathbf{c}$*, the following two implications hold:*

- *if* $\mathbf{a} < \mathbf{b}$*, then* $\mathbf{a} + \mathbf{c} < \mathbf{b} + \mathbf{c}$*;*
- *if* $\mathbf{a} \sim \mathbf{b}$*, then* $\mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{c}$.

Now, we are ready to formulate our main result.

Proposition. *For every natural scale-invariant additive linear pre-order on the set of all possible intervals, there exists a number* α_H *for which the pre-order has one of the following three forms:*

- $[\underline{a}, \overline{a}] < [\underline{b}, \overline{b}]$ *if and only if*
$$\alpha_H \cdot \overline{a} + (1 - \alpha_H) \cdot \underline{a} < \alpha_H \cdot \overline{b} + (1 - \alpha_H) \cdot \underline{b}; \quad (1)$$
- *for* $\alpha_H < 1$, $\mathbf{a} = [\underline{a}, \overline{a}] < \mathbf{b} = [\underline{b}, \overline{b}]$ *if and only if*
  - *either we have an inequality (1)*
  - *or we have an equality*
$$\alpha_H \cdot \overline{a} + (1 - \alpha_H) \cdot \underline{a} = \alpha_H \cdot \overline{b} + (1 - \alpha_H) \cdot \underline{b}, \quad (2)$$
    and $\mathbf{a}$ *is wider than* $\mathbf{b}$*, i.e.,* $\overline{a} - \underline{a} > \overline{b} - \underline{b}$*;*
- *for* $\alpha_H > 0$, $\mathbf{a} = [\underline{a}, \overline{a}] < \mathbf{b} = [\underline{b}, \overline{b}]$ *if and only if:*
  - *either we have the inequality (1)*
  - *or we have the equality (2) and* $\mathbf{a}$ *is narrower than* $\mathbf{b}$*, i.e.,* $\overline{a} - \underline{a} < \overline{b} - \underline{b}$.

Vice versa, for each $\alpha_H \in [0, 1]$*, all three relations are natural scale-invariant consistent pre-orders on the set of all possible intervals.*

Discussion.

- The first relation describes a risk-neutral decision maker, for whom all intervals with the same Hurwicz equivalent value are indeed equivalent.
- The second relation describes a risk-averse decision maker, who from all the intervals with the same Hurwicz equivalent value selects the one which is the narrowest, i.e., for which the risk is the smallest.

- Finally, the third relation describes a risk-prone decision maker, who from all the intervals with the same Hurwicz equivalent value selects the one which is the widest, i.e., for which the risk is the largest.

Interesting fact. All three cases can be naturally described in yet another way: in terms of the so-called *non-standard analysis* (see, e.g., [1, 3, 6, 7]), where, in addition to usual ("standard") real numbers, we have *infinitesimal* real numbers, i.e., e.g., objects ε which are positive but which are smaller than all positive standard real numbers.

We can perform usual arithmetic operations on all the numbers, standard and others ("non-standard"). In particular, for every real number x, we can consider non-standard numbers $x+\varepsilon$ and $x-\varepsilon$, where $\varepsilon > 0$ is a positive infinitesimal number – and, vice versa, every non-standard real number which is bounded from below and from above by some standard real numbers can be represented in one of these two forms.

From the above definition, we can conclude how to compare two non-standard numbers obtained by using the same infinitesimal $\varepsilon > 0$, i.e., to be precise, how to compare the numbers $x+k\cdot\varepsilon$ and $x'+k'\cdot\varepsilon$, where x, k, x', and k' are standard real numbers. Indeed, the inequality

$$x+k\cdot\varepsilon < x'+k'\cdot\varepsilon \tag{3}$$

is equivalent to

$$(k-k')\cdot\varepsilon < (x'-x).$$

- If $x' > x$, then this inequality is true since any infinitesimal number (including the number $(k-k')\cdot\varepsilon$) is smaller than any standard positive number – in particular, smaller than the standard real number $x'-x$.
- If $x' < x$, then this inequality is not true, because we will then similarly have $(k'-k)\cdot\varepsilon < (x-x')$, and thus, $(k-k')\cdot\varepsilon > (x'-x)$.
- Finally, if $x = x'$, then, since $\varepsilon > 0$, the above inequality is equivalent to $k < k'$.

Thus, the inequality (3) holds if and only if:

- either $x < x'$,
- or $x = x'$ and $k < k'$.

If we use non-standard numbers, then all three forms listed in the Proposition can be described in purely Hurwicz terms:

$$(\mathbf{a} = [\underline{a},\overline{a}] < \mathbf{b} = [\underline{b},\overline{b}]) \Leftrightarrow (\alpha_{NS}\cdot\overline{a}+(1-\alpha_{NS})\cdot\underline{a} < \alpha_{NS}\cdot\overline{b}+(1-\alpha_{NS})\cdot\underline{b}), \tag{4}$$

for some $\alpha_{NS} \in [0,1]$; the only difference from the traditional Hurwicz approach is that now the value α_{NS} can be non-standard. Indeed:

- If α_{NS} is a standard real number, then we get the usual Hurwicz ordering – which is the first form from the Proposition.

- If α_{NS} has the form $\alpha_{NS} = \alpha_H - \varepsilon$ for some standard real number α_H, then the inequality (4) takes the form

$$(\alpha_H - \varepsilon) \cdot \overline{a} + (1 - (\alpha_H - \varepsilon)) \cdot \underline{a} < (\alpha_H - \varepsilon) \cdot \overline{b} + (1 - (\alpha_H - \varepsilon)) \cdot \underline{b},$$

i.e., separating the standard and infinitesimal parts, the form

$$(\alpha_H \cdot \overline{a} + (1 - \alpha_H) \cdot \underline{a}) - (\overline{a} - \underline{a}) \cdot \varepsilon < (\alpha_H \cdot \overline{b} + (1 - \alpha_H) \cdot \underline{b}) - (\overline{b} - \underline{b}) \cdot \varepsilon.$$

Thus, according to the above description of how to compare non-standard numbers, we conclude that for $\alpha_{NS} = \alpha_H - \varepsilon$, we have $\mathbf{a} < \mathbf{b}$ if and only if:

– either we have the inequality (1)
– or we have the equality (2) and **a** is wider than **b**, i.e., $\overline{a} - \underline{a} > \overline{b} - \underline{b}$.

This is exactly the second form from our Proposition.

- Finally, if α_{NS} has the form $\alpha_{NS} = \alpha_H + \varepsilon$ for some standard real number α_H, then the inequality (4) takes the form

$$(\alpha_H + \varepsilon) \cdot \overline{a} + (1 - (\alpha_H + \varepsilon)) \cdot \underline{a} < (\alpha_H + \varepsilon) \cdot \overline{b} + (1 - (\alpha_H + \varepsilon)) \cdot \underline{b},$$

i.e., separating the standard and infinitesimal parts, the form

$$(\alpha_H \cdot \overline{a} + (1 - \alpha_H) \cdot \underline{a}) + (\overline{a} - \underline{a}) \cdot \varepsilon < (\alpha_H \cdot \overline{b} + (1 - \alpha_H) \cdot \underline{b}) + (\overline{b} - \underline{b}) \cdot \varepsilon.$$

Thus, according to the above description of how to compare non-standard numbers, we conclude that for $\alpha_{NS} = \alpha_H + \varepsilon$, we have $\mathbf{a} < \mathbf{b}$ if and only if:

– either we have the inequality (1)
– or we have the equality (2) and **a** is narrower than **b**, i.e., $\overline{a} - \underline{a} < \overline{b} - \underline{b}$.

This is exactly the third form from our Proposition.

3 Proof

1°. Let us start with the same interval $[0,1]$ as in the above derivation of the Hurwicz criterion.

1.1°. If the interval $[0,1]$ is equivalent to some real number α_H – i.e., strictly speaking, to the corresponding degenerate interval $[0,1] \sim [\alpha_H, \alpha_H]$, then, similarly to that derivation, we can conclude that every interval $[\underline{a}, \overline{a}]$ is equivalent to its Hurwicz equivalent value $\alpha_H \cdot \overline{a} + (1 - \alpha_H) \cdot \underline{a}$. Here, because of naturalness, we have $\alpha_H \in [0,1]$.

This is the first option from the formulation of our Proposition.

1.2°. To complete the proof, it is thus sufficient to consider the case when the interval $[0,1]$ is *not* equivalent to any real number. Since we consider a linear pre-order, this means that for every real number r, the interval $[0,1]$ is either smaller or larger.

- If for some real number a, we have $a < [0,1]$, then, due to transitivity and naturalness, we have $a' < [0,1]$ for all $a' < a$.
- Similarly, if for some real number b, we have $[0,1] < b$, then we have $[0,1] < b'$ for all $b' > b$.

Thus, there is a threshold value

$$\alpha_H = \sup\{a : a < [0,1]\} = \inf\{b : [0,1] < b\}$$

such that:

- for $a < \alpha_H$, we have $a < [0,1]$, and
- for $a > \alpha_H$, we have $[0,1] < a$.

Because of naturalness, we have $\alpha_H \in [0,1]$.

Since we consider the case when the interval $[0,1]$ is not equivalent to any real number, we this have either $[0,1] < \alpha_H$ or $\alpha_H < [0,1]$.

Let us first consider the first option.

2°. In the first option, due to scale-invariance and additivity with $\mathbf{c} = [\underline{a}, \underline{a}]$, similarly to the above derivation of the Hurwicz criterion, for every interval $[\underline{a}, \overline{a}]$, we have:

- when $a < \alpha_H \cdot \overline{a} + (1-\alpha_H) \cdot \underline{a}$, then $a < [\underline{a}, \overline{a}]$; and
- when $a \geq \alpha_H \cdot \overline{a} + (1-\alpha_H) \cdot \underline{a}$, then $[\underline{a}, \overline{a}] \leq a$.

Thus, if the Hurwicz equivalent value $u_H(\mathbf{a})$ of a non-degenerate interval $\mathbf{a}$ is smaller than the Hurwicz equivalent value $u_H(\mathbf{a})$ of a non-degenerate interval $\mathbf{b}$, we can conclude that

$$\mathbf{a} < \frac{u_H(\mathbf{a}) + u_H(\mathbf{b})}{2} < \mathbf{b}$$

and hence, that $\mathbf{a} < \mathbf{b}$. So, to complete the description of the desired linear pre-order, it is sufficient to be able to compare the intervals with the same Hurwicz equivalent value.

3°. One can easily check that for every $k > 0$, the Hurwicz equivalent value of the interval $[-k \cdot \alpha_H, k \cdot (1-\alpha_H)]$ is 0.

Thus, in the first option, we have $[-k \cdot \alpha_H, k \cdot (1-\alpha_H)] < 0$. So, for every $k' > 0$, by using additivity with $\mathbf{c} = [-k' \cdot \alpha_H, k' \cdot (1-\alpha_H)]$, we conclude that

$$[-(k+k') \cdot \alpha_H, (k+k') \cdot (1-\alpha_H)] < [-k \cdot \alpha_H, k \cdot (1-\alpha_H)].$$

Hence, for two intervals with the same Hurwicz equivalent value 0, the narrower one is better.

By applying additivity with $\mathbf{c}$ equal to Hurwicz value, we conclude that the same is true for all possible Hurwicz equivalent values.

This is the second case in the formulation of our proposition.

4°. Similarly to Part 2 of this proof, in the second option, when $\alpha_H < [0,1]$, we can also conclude that if the Hurwicz equivalent value $u_H(\mathbf{a})$ of a non-degenerate interval **a** is smaller than the Hurwicz equivalent value $u_H(\mathbf{a})$ of a non-degenerate interval **b**, then $\mathbf{a} < \mathbf{b}$.

Then, similarly to Part 3 of this proof, we can prove that for two intervals with the same Hurwicz equivalent value, the wider one is better.

This is the third option as described in the Proposition.

The Proposition is thus proven.

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