Experimental Determination of Mechanical Properties Is, In General, NP-Hard – Unless We Measure Everything

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Abstract

When forces are applied to different parts of a construction, they cause displacements. In practice, displacements are usually reasonably small. In this case, we can safely ignore quadratic and higher order terms in the corresponding dependence and assume that the forces linear depend on displacements. The coefficients of this linear dependence determine the mechanical properties of the construction and thus, need to be experimentally determined. In the ideal case, when we measure the forces and displacements at all possible locations, it is easy to find the corresponding coefficients: it is sufficient to solve the corresponding system of linear equations. In practice, however, we only measure displacements and forces at *some* locations. We show that in this case, the problem of determining the corresponding coefficients becomes, in general, NP-hard.

1 Formulation of the Problem

Linear elasticity: a brief reminder. A force applied to a rubber band extends it or curves it. In general, a force applied to different parts of an elastic body changes the mutual location of its points. Once we know the forces applied at different locations, we can determine the deformations – and, vice versa, we can determine the forces once we know all the deformations.

In general, the dependence on forces f_α at different locations α on different displacement ε_β is non-linear. However, when displacements are small, we can ignore terms quadratic or higher order in terms of ε_β and thus safely assume that the dependence of each force component f_α on all the components ε_β of displacements at different locations β is linear.

Taking into account that in the absence of forces, there is no displacement, we conclude that $f_\alpha = \sum_\beta K_{\alpha,\beta} \cdot \varepsilon_\beta$ for some coefficients $K_{\alpha,\beta}$. These coefficients $K_{\alpha,\beta}$ describe the mechanical properties of the body.

It is therefore desirable to experimentally determine these coefficients.

Ideal case. In the ideal case, we measure displacements ε_β and forces f_α at all possible locations.

Each such measurement results in an equation $f_\alpha = \sum_\beta K_{\alpha,\beta} \cdot \varepsilon_\beta$ which is linear in terms of the unknowns $K_{\alpha,\beta}$. Thus, after performing sufficiently many measurements, we get an easy-to-solve system of linear equations that enables us to find the values $K_{\alpha,\beta}$.

In practice, we only measure some values. In reality, we only measure displacements and forces at some locations – i.e., we know only some values f_α and ε_β.

In this case, since both $K_{\alpha,\beta}$ and some values ε_β are unknown, the corresponding system of equations becomes quadratic.

After sufficiently many measurements, we may still uniquely determine $K_{\alpha,\beta}$, but the reconstruction is more complex.

How complex: what we prove. How complex is the corresponding computational problem?

In this paper, we prove that the corresponding reconstruction problem is, in general, NP-hard. This means that, if – as most computer scientists believe – NP $\neq$ P, no feasible algorithm is possible that would always reconstruct the mechanical properties $K_{\alpha,\beta}$ based on the experimental results; see, e.g., [1, 2].

We will prove NP-hardness even for the problem of checking whether a given value of K_{α_0,β_0} for some α_0 and β_0 is consistent with the observations.

2 Definitions and the Main Result

From the computational viewpoint, the above problem can be formulated as follows.

Definition 1. *Let K be a natural number. This number will be called* the number of experiments. *By a* problem of experimentally determining mechanical properties*, we mean the following problem.*

- *We know that for every k from 1 to K, we have $f_\alpha^{(k)} = \sum_\beta K_{\alpha,\beta} \cdot \varepsilon_\beta^{(k)}$ for some values $f_\alpha^{(k)}$ and $\varepsilon_\beta^{(k)}$.*
- *For each k, we know some of the values $f_\alpha^{(k)}$ and $\varepsilon_\beta^{(k)}$.*
- *We need to check if for given α_0, β_0, and K_0, we can have $K_{\alpha_0,\beta_0} = K_0$.*

Proposition. *The problem of experimentally determining mechanical properties is NP-hard.*

Proof.

1°. By definition, NP-hard means that all the problems from a certain class NP can be reduced to this problem; see, e.g., [1, 2]. It is known that the following *subset sum* problem is NP-hard:

- given $m + 1$ natural numbers $s_1, \ldots, s_m, S$,
- check whether it is possible to find the values $x_i \in \{0, 1\}$ for which

$$\sum_{i=1}^{m} s_i \cdot x_i = S$$

  (in other words, check whether it is possible to find a subset of the values $s_1, \ldots, s_m$ whose sum is equal to the given value S).

The fact that the subset sum problem is NP-hard means that every problem from the class NP can be reduced to this problem. So, if we reduce the subset problem to our problem, that would mean, by transitivity of reduction, that every problem from the class NP can be reduced to our problem as well – i.e., that our problem is indeed NP-hard.

This is exactly how we will construct this proof – by showing that the subset sum problem can be reduced to our problem.

2°. Let $s_1, \ldots, s_m, S$ be the values that describe an instance of the subset sum problem. Let us reduce it to the following instance of our problem. In this instance, we have $2m + 1$ variables $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_m, \varepsilon_{m+1}, \ldots, \varepsilon_{2m}$. We also have $m + 1$ different values f_α, $\alpha = 0, 1, \ldots, m$.

3°. In the first series of experiments $k = 1, \ldots, m$, for each $i = 1, \ldots, m$, we have $\varepsilon_i^{(i)} = 1$, $\varepsilon_{m+i}^{(i)} = -1$, and $\varepsilon_j^{(i)} = 0$ for all $j \neq i$. The only value of f_α that we measure in each of these experiments is the value $f_0^{(i)} = 0$.

From the corresponding equation

$$0 = f_0^{(i)} = \sum_\beta K_{0,\beta} \cdot \varepsilon_\beta^{(i)} = K_{0,i} - K_{0,m+i},$$

we conclude that

$$K_{0,m+i} = K_{0,i}. \quad (1)$$

4°. In the second series of experiments $k = m + 1, \ldots, m + i, \ldots, 2m$, where $i = 1, \ldots, m$, for each $k = m + i$, we measure the values $\varepsilon_j^{(m+i)} = 0$ for all $j \neq k$, and we measure the values $f_0^{(m+i)} = f_i^{(m+i)} = 1$.

From the corresponding equations, we conclude that $1 = K_{0,m+i} \cdot \varepsilon_{m+i}^{(m+i)}$ and $1 = K_{i,m+i} \cdot \varepsilon_{m+i}^{(m+i)}$. We do not know the value $\varepsilon_{m+i}^{(m+i)}$, but we can find it from the first equation and substitute into the second one. As a result, we conclude that

$$K_{0,m+i} = K_{i,m+i}. \tag{2}$$

Combining equalities (1) and (2), we conclude that

$$K_{0,i} = K_{i,m+i}. \tag{3}$$

5°. In the third series of experiments $k = 2m + i$, $i = 1, \ldots, m$, for each i, we measure $\varepsilon_i^{(2m+i)} = 1$, $\varepsilon_j^{(2m+i)} = 0$ for all other j, and we measure $f_i^{(2m+i)} = 1$.

The corresponding equation implies that

$$K_{i,i} = 1. \tag{4}$$

6°. In the fourth series of experiments $k = 3m + i$, $i = 1, \ldots, m$, we measure the values $\varepsilon_{m+i}^{(3m+i)} = -1$ and $\varepsilon_j^{(3m+i)} = 0$ for all j which are different from i and from $m + i$. We also measure the values $f_0^{(3m+i)} = f_i^{(3m+i)} = 0$.

In this case, the corresponding equations lead to

$$K_{0,i} \cdot \varepsilon_i^{(3m+i)} - K_{0,m+i} = 0 \tag{5}$$

and

$$K_{i,i} \cdot \varepsilon_i^{(3m+i)} - K_{i,m+i} = 0. \tag{6}$$

Since, due to (4), we have $K_{i,i} = 1$, the equation (6) simply means that $\varepsilon_i^{(3m+i)} = K_{i,m+i}$. Due to formula (3), this implies that $\varepsilon_i^{(3m+i)} = K_{0,i}$. Substituting this expression for $\varepsilon_i^{(3m+i)}$ into the equation (5) and taking into account that, due to (1), we have $K_{0,m+i} = K_{0,i}$, we conclude that $K_{0,i}^2 - K_{0,i} = 0$.

From $K_{0,i} \cdot (K_{0,i} - 1) = 0$, we conclude that either $K_{0,i} = 0$ or $K_{0,i} = 1$. Thus, for each i from 1 to m, we have $K_{0,i} \in \{0, 1\}$.

7°. The final, fifth series of experiments consists of only one experiment $k = 4m + 1$. In this experiment, we measure the values $\varepsilon_0^{(4m+1)} = -S$,

$$\varepsilon_1^{(4m+1)} = s_1, \ldots, \varepsilon_m^{(4m+1)} = s_m,$$

and $\varepsilon_{m+i}^{(4m+1)} = 0$ for all $i = 1, \ldots, m$. We also measure $f_0^{(4m+1)} = 0$.

We want to check whether it is possible that $K_{0,0} = 1$.

For $K_{0,0} = 1$, the corresponding equation takes the form

$$-S + K_{0,1} \cdot s_1 + \ldots + K_{0,m} \cdot s_m = 0,$$

i.e., the form

$$K_{0,1} \cdot s_1 + \ldots + K_{0,m} \cdot s_m = S \tag{7}$$

for some values $K_{0,i} \in \{0, 1\}$.

One can easily see that:

- If the original instance of the subset sum problem has a solution $x_i \in \{0, 1\}$, then the above equality holds for $K_{0,i} = x_i$.
- Vice versa, if there exist values $K_{0,i}$ that satisfy the formula (5), then the values $x_i = K_{0,i}$ solve the original subset sum problem.

Thus, we indeed have a reduction – and hence, our problem is indeed NP-hard.

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