Computing with Words – When Results Do Not Depend on the Selection of the Membership Function

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Abstract

Often, we need to transform natural-language expert knowledge into computer-understandable numerical form. One of the most successful ways to do it is to use fuzzy logic and membership functions. The problem is that membership functions are subjective. It is therefore desirable to look for cases when the results do not depend on this subjective choice. In this paper, after describing a known example of such a situation, we list several other examples where the results do not depend on the subjective choice of a membership function.

1 Formulation of the Problem

Often, we need to transform natural-language expert knowledge into computer-understandable numerical form. One of the most successful ways to do it is to use fuzzy logic; see, e.g., [1, 2, 3, 4, 5, 6].

In fuzzy logic, each imprecise property like "small" is described by a *membership function* that assigns, to each possible value x, a degree $\mu(x)$ to which x is, e.g., small.

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2 Continuity: A Known Example

Intuitive notion of continuity. What is continuity? There is a mathematical definition, but what is the intuitive notion of continuity?

Intuitively, continuity means that if x' is close to x, then $y' = f(x')$ should be close to $y = f(x)$. In other words, this means that if the difference $x' - x$ between the x-values is small, then the difference $f(x') - f(x)$ between the y-values should also be small.

How can we formalize if-then statements. If a statement S implies the statement S', this means that our degree of confidence in a statement S' should be larger than or equal to our degree of confidence in S. Indeed, every time we believe in S, we should also, because of the implication, believe in S' – and we may also have additional cases when we believe in S' but not in S.

The notion of small. The notion "small" for x-values Δx is described by the corresponding membership function $\mu^{\rm x}_{\rm small}(\Delta x)$. Specifically, our degree of believe that the difference $x' - x$ is small is equal to $\mu^{\rm x}_{\rm small}(x' - x)$.

Intuitively, a value x is small if and only if its opposite $-\Delta x$ is also small. Thus, the degree of believe that a negative number Δx is small is equal to the degree of confidence that its absolute value $|\Delta x|$ is small. So, in all cases, we have $\mu^{\rm x}_{\rm small}(\Delta x) = \mu^{\rm x}_{\rm small}(|\Delta x|)$. Thus, our degree of confidence that the difference $x' - x$ is small is equal to $\mu^{\rm x}_{\rm small}(|x' - x|)$.

Similarly, our degree of confidence that the difference $f(x') - f(x)$ between the y-values is small is equal to $\mu^{\rm y}_{\rm small}(|f(x') - f(x)|)$, where $\mu^{\rm y}_{\rm small}(\Delta y)$ is a membership function describing when y-values are small.

So, what is intuitive continuity. In view of the above, continuity means that for all x and x', we have

$$\mu^{\rm y}_{\rm small}(|x' - x|) \geq \mu^{\rm x}_{\rm small}(|x' - x|).$$

Scales for x and y are, in general, different. The two membership functions $\mu^{\rm x}_{\rm small}(\Delta x)$ and $\mu^{\rm y}_{\rm small}(\Delta y)$ corresponding to x- and y-values describe the same notion of smallness, but it may correspond to different scales. To take this into account, let us assume that one y-unit is equivalent to K x-units.

This means that $\mu^{\rm y}_{\rm small}(z) = \mu^{\rm x}_{\rm small}(K \cdot z)$ and thus,

$$\mu^{\rm x}_{\rm small}(K \cdot (f(x') - f(x)) \geq \mu^{\rm x}_{\rm small}(x' - x). \tag{1}$$

So, what is intuitive continuity? The larger the value z, the less confident we are that this value is small. Thus, the function $\mu_{\rm small}(z)$ is monotonically decreasing. Hence, the inequality (1) implies that $K\cdot|f(x')-f(x)|\le|x-x'|$ and, therefore,

$$|f(x')-f(x)|\le K^{-1}\cdot|x'-x|.$$

Thus, the common sense continuity leads to what is known in mathematics as the Lipschitz condition.

Let us describe this is precise terms. Let us describe the above argument in precise terms.

Definition 1. *By a* membership function corresponding to x-small*, we mean an even function* $\mu^{\rm x}_{\rm small}(-x)=\mu^{\rm x}_{\rm small}(x)$ *which is strictly decreasing for* $x\ge 0$. *For each real number* x*, by the* degree of confidence that x is small*, we mean the value* $\mu^{\rm x}_{\rm small}(x)$.

Definition 2. *By a* membership function corresponding to y-small*, we mean a function* $\mu^{\rm y}_{\rm small}(y)=\mu^{\rm x}_{\rm small}(K\cdot y)$*, for some constant* $K>0$. *For each real number* y*, by the* degree of confidence that y is small*, we mean the value* $\mu^{\rm y}_{\rm small}(y)$.

Definition 3. *For every two numbers* x *and* x'*, by the* degree of confidence that x' is close to x*, we mean the degree of confidence that the difference* $x'-x$ *is* x*-small.*

Definition 4. *For every two numbers* y *and* y'*, by the* degree of confidence that y' is close to y*, we mean the degree of confidence that the difference* $y'-y$ *is small.*

Definition 5. *We say that a statement* S implies *the statement* S' *(or, equivalently, that* whenever S then S'*) if the degree of confidence in* S *is smaller than or equal to the degree of confidence in* S'.

Definition 6. *We say that a function* $f(x)$ *is* intuitively continuous *if whenever* x' *is close to* x*, the value* $f(x')$ *is close to* $f(x)$.

Definition 7. *We say that a function* $f(x)$ *is* Lipschitz continuous *if there exists a constant* $C>0$ *such that for every* x *and for every* x'*, we have*

$$|f(x')-f(x)|\le C\cdot|x'-x|.$$

Proposition 1. *A function is intuitively continuous if and only if it is Lipschitz continuous with* $C=K^{-1}$.

Comment 1. The proof was, in effect, given above.

Comment 2. For a differentiable function, Lipshitz continuity is equivalent to the upper bound on the derivative: $|f'(x)|\le C$ for all x.

Comment 3. In the above text, we considered the two values x and x' to be close if the difference $x'-x$ between them is close to 0. This makes sense if the range of possible values is reasonably small. However, when the range is large, then it stops making sense: indeed, intuitively, 1010 is close to 1001 but 10 is not close to 1. In such cases, a better description of intuitive closeness is that the ratio x'/x of two numbers is close to 1. This is equivalent to saying that the logarithm $\ln(x'/x) = \ln(x') - \ln(x)$ of this ratio is close to 0.

Thus, this idea is equivalent to applying the above result to new variables $X \stackrel{\text{def}}{=} \ln(x)$ and $Y \stackrel{\text{def}}{=} \ln(y)$. So, we conclude that in this log-log scale, we get a Lipschitz function. In particular, if we substitute $X = \ln(x)$ and $Y = \ln(y)$ into the condition $\left|\dfrac{dY}{dX}\right| \le C = K^{-1}$, we get, for the original dependence $y = f(x)$, the condition $|f'(x)| \le C \cdot \dfrac{f(x)}{x}$.

3 First New Example

Idea. What if we have not a function but, more generally, a relation R between x and y – i.e., a set of pairs $R \subseteq X \times Y$? In this case, we have a similar result:

Definition 8. *We say that a relation R is* intuitively continuous *if for every two pairs $(x, y) \in R$, $(x', y') \in R$, whenever x' is close to x, the value y' is also close to y.*

Proposition 2. *If a relation is intuitively continuous, then it is a function.*

Proof. To prove that the relation is a function, we need to prove that for every x, there exists only one y for which $(x, y) \in R$. In other words, we need to prove that if $(x, y) \in R$ and $(x, y') \in R$, then $y = y'$.

Indeed, in this case, from Definition 8, it follows that $\mu^{\rm y}_{\rm small}(y' - y) \ge \mu^{\rm x}_{\rm small}(x - x) = \mu^{\rm x}_{\rm small}(0)$. By definition of y-closeness, we have $\mu^{\rm y}_{\rm small}(y' - y) = \mu^{\rm x}_{\rm small}(K \cdot (y' - y))$, so we get $\mu^{\rm x}_{\rm small}(K \cdot (y' - y)) \ge \mu^{\rm x}_{\rm small}(0)$. Since the function $\mu^{\rm x}_{\rm small}(x)$ is even and strictly decreasing for $x \ge 0$, this implies that $|y' - y| \le 0$, thus indeed $y = y'$. The proposition is proven.

Corollary. *Every intuitively continuous relation is a Lipschitz continuous function, with $C = K^{-1}$.*

Comment. This corollary follows from Propositions 1 and 2.

4 Second New Example

Idea. What is the dependence of y on x and x on y are both intuitively continuous? In other words, what if x' is close to x if and only if $f(x')$ is close to $f(x)$?

Proposition 3. *For a function f, the following two conditions are equivalent to each other:*

- *the function $f(x)$ and the inverse function $f^{-1}(y)$ are both intuitively continuous;*

- *the function $f(x)$ is linear $f(x) = a + b \cdot x$, with $b = \pm K^{-1}$.*

Proof.

1°. It is easy to see that for a linear function $a + b \cdot x$, with $b = \pm K$, both this function and its inverse are Lipschitz continuous with the proper coefficients and thus, are both intuitively continuous.

2°. Vice versa, let us assume that both the function $f(x)$ and its inverse are intuitively continuous – and thus, Lipschitz continuous. Then, we conclude that for every two values x and x', we have $|f(x') - f(x)| \le K^{-1} \cdot |x' - x|$ and $K^{-1} \cdot |x' - x| \le |f(x') - f(x)|$. So, we get $|f(x') - f(x)| = K^{-1} \cdot |x' - x|$ for all x and x'.

In particular, for $x = 0$ and $x' = 1$, we get $|f(1) - f(0)| = K^{-1}$. So, either $f(1) - f(0) = K^{-1}$ or $f(1) - f(0) = -K^{-1}$. Let us consider these two cases one by one.

2.1°. Let us first consider the case when $f(1) - f(0) = K^{-1}$, so that $f(1) = f(0) + K^{-1}$. Let us prove that in this case, for all x, we have $f(x) = f(0) + K^{-1} \cdot x$.

Indeed, for every x, we have $|f(x) - f(0)| = K^{-1} \cdot |x|$, so we have:

- either $f(x) - f(0) = K^{-1} \cdot x$ and thus, $f(x) = f(0) + K^{-1} \cdot x$,
- or $f(x) - f(0) = -K^{-1} \cdot x$ and thus, $f(x) = f(0) - K^{-1} \cdot x$.

We want to prove that for $x \ne 0$, the formula is not possible (for $x = 0$, both formulas lead to the same value $f(x) = f(0)$).

Indeed, if $f(x) = f(0) - K^{-1} \cdot x$, then

$$f(x) - f(1) = (f(0) - K^{-1} \cdot x) - (f(0) + K^{-1}) = -K^{-1} \cdot (x + 1),$$

and thus, $|f(x) - f(1)| = K^{-1} \cdot |x + 1|$ while we should have $|f(x) - f(1)| = K^{-1} \cdot |x - 1|$. So, we have $|x + 1| = |x - 1|$, and thus,

- either $x + 1 = x - 1$, which leads to $1 = -1$ and is, thus, impossible,
- or $x + 1 = -(x - 1)$, i.e., $x + 1 = -x + 1$ and thus, $x = 0$.

So, for $x \ne 0$, we indeed cannot have $f(x) = f(0) - K^{-1} \cdot x$ and thus, we indeed have $f(x) = f(0) + K^{-1} \cdot x$.

2.2°. Similarly, we can prove our result in the case when $f(1) - f(0) = -K^{-1}$. So, in both cases, the proposition is proven.

Comment. This result may explain the ubiquity of linear dependencies.

5 Third New Example

Idea. What if the function $f(x)$ is growing? From the purely mathematical viewpoint, it means that if $x' > x$, then $f(x') > f(x)$. However, from the common sense viewpoint, if an economy grew by 0.1% in a year, we will not say that it is growing: we will say that the economy is stagnating. From the common sense viewpoint, growing means that if x' is much larger than x, then $f(x')$ should be much larger than $f(x)$. Let us formalize this intuitive notion.

Definition 9. *By a* membership function corresponding to x-much larger*, we mean a function $\mu^{\rm x}_{\gg}(x)$ which is equal to 0 for $x \le 0$ and is strictly increasing for $x \ge 0$. For each real number x, by the* degree of confidence that x is much larger than 0*, we mean the value $\mu^{\rm x}_{\gg}(x)$.*

Definition 10. *By a* membership function corresponding to y-much larger*, we mean a function $\mu^{\rm y}_{\gg}(y) = \mu^{\rm x}_{\gg}(K \cdot y)$, for some constant $K > 0$. For each real number y, by the* degree of confidence that y is much larger than 0*, we mean the value $\mu^{\rm y}_{\gg}(y)$.*

Definition 11. *For every two numbers x and x', by the* degree of confidence that x' is much larger than x*, we mean the degree of confidence that the difference $x' - x$ is x-much larger than 0.*

Definition 12. *For every two numbers y and y', by the* degree of confidence that y' is much larger than y*, we mean the degree of confidence that the difference $y' - y$ is much larger than 0.*

Definition 13. *We say that a function $f(x)$ is* intuitively growing *if whenever x' is much larger than x, the value $f(x')$ is much larger than $f(x)$.*

Proposition 4. *A function $f(x)$ is intuitively growing if and only if for every pair $x < x'$, we have $f(x') - f(x) \ge K^{-1} \cdot (x' - x)$.*

Proof. By definition, intuitively growing means that for all $x' > x$, we have $\mu^{\rm y}_{\gg}(f(x') - f(x)) \ge \mu^{\rm x}{\gg}(x' - x)$, i.e., equivalently, that $\mu^{\rm x}_{\gg}(K \cdot (f(x') - f(x))) \ge \mu^{\rm x}_{\gg}(x' - x)$. Since the function $\mu^{\rm x}_{\gg}(x)$ is strictly increasing, this inequality is equivalent to $K \cdot (f(x') - f(x)) \ge x' - x$, and thus, equivalent to the desired inequality $f(x') - f(x) \ge K^{-1} \cdot (x' - x)$.

The proposition is proven.

Comment. For a differentiable function, the above condition is equivalent to the lower bound on the derivative: $f'(x) \ge K^{-1}$ for all x.

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