Between Dog and Wolf: A Continuous Transition from Fuzzy to Probabilistic Estimates

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***Abstract*—Often, we use original expert estimates to compute estimates of related quantities. In many practical situations, it is desirable to know how accurate is the resulting estimate. There are many techniques for computing this accuracy: we can use simple probabilistic ideas and we can use simple fuzzy ideas. Strangely enough, these two reasonable techniques lead to drastically different results. Which of them is correct? Our practical tests show that none of these two methods is perfect: probabilistic approach usually underestimates uncertainty, while the fuzzy approach overestimates it. This looks similar to many cases that motivated Zadeh to promote the idea of soft computing – a combination of different uncertainty techniques. To get a more adequate combination technique, we analyzed the general problem of combining accuracy estimates and came up with a 1-parametric family of techniques that contains probabilistic and fuzzy as particular cases – and that indeed works better on several practical examples that each of the original two techniques.**

***Index Terms*—uncertainty estimation, probabilistic approach, fuzzy approach, scale-invariance, geophysical applications**

I. Introduction

Computations based on expert estimates: a typical situation. In many practical situations, we have expert estimates $\widetilde{x}_1, \ldots, \widetilde{x}_n$ of several quantities $x_1, \ldots, x_n$.

Based on these estimates, we often estimate the values of other quantities y that depend on x_i in a known way, i.e., for which we know the dependence $y = f(x_1, \ldots, x_n)$. Namely, as the desired estimate for y, it is natural to take the value

$$\widetilde{y} = f\left(\widetilde{x}_1, \ldots, \widetilde{x}_n\right).$$

For example, if we have a reasonable estimate x_1 for the distance that a car travelled, and an estimate x_2 for the time it took the car to travel, we can divide these two values and thus provide the estimates $\widetilde{y} = \dfrac{\widetilde{x}_1}{\widetilde{x}_2}$ for the car's speed $y = \dfrac{x_1}{x_2}$.

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In many situations, accuracy estimation is important. In many practical situations, it is important to know the accuracy of the resulting estimate $\widetilde{y}$; see, e.g., [13].

For example, in economics, if we predict the nearest-future change in stock prices, it is important to know how accurate is this prediction: using an inaccurate estimate can lead to huge money losses.

Similarly, in geophysics: if, based on the expert estimates, we come up with a reasonable expected amount of oil $\widetilde{y}$ in a given area, the question is – how accurate is this estimate? If this estimate is reasonably accurate, then it makes sense to invest in this oil field. However, if the estimate $\widetilde{y}$ is not very accurate, maybe it is better to perform some additional measurements and/or ask additional experts, to avoid sinking millions of dollars into an unproductive area.

In medicine, we want to make sure that the estimate correctly describes the state of the patient's health and that, therefore, the cure is right for the given patient – otherwise, by prescribing a wrong treatment, we can make the disease worse or even lose the patient.

Resulting computational problem. To estimate the accuracy of the combined estimate $\widetilde{y}$, we need to know how accurate are the original estimates $\widetilde{x}_1, \ldots, \widetilde{x}_n$.

Usually, for each of these estimates $\widetilde{x}_i$, we know a number Δ_i that describes its accuracy, i.e., that describes the order of the possible approximation error $\Delta x_i \stackrel{\text{def}}{=} \widetilde{x}_i - x_i$: Δx_i is approximately of the same order as Δ_i.

Based on the values Δ_i describing the accuracy of each original estimate $\widetilde{x}_i$, we would like to compute a number Δ describing the accuracy of the resulting estimate $\widetilde{y}$.

How this problem is solved now: general idea. There are many techniques for solving the above problem. These techniques depend on how exactly the value Δ_i relates to the approximation error.

Case of interval uncertainty. This number Δ_i can be the upper bound on the possible values of the approximation error. This is the case of *interval uncertainty*; see, e.g., [4], [8], [10], [13].

Case of probabilistic uncertainty. The number Δ_i can be the mean squared value of the approximation error, or the most probable value of this error. These are the two cases of *probabilistic uncertainty*; see, e.g., [13], [14]. The corresponding methods are described in Section 3.

Case of fuzzy uncertainty. The number Δ_i can simply be an expert's estimate for the approximation error; this is the case of *fuzzy uncertainty*; see, e.g., [1], [5], [9], [11], [12], [15]. The corresponding methods are described in Section 4.

Remaining challenges. At first glance, the corresponding computational problem seems straightforward: there are reasonable approaches for solving it. For example:

- we can use simply probabilistic ideas, or
- we can use simple fuzzy ideas.

But here lies the challenge: as we will see in Sections 3 and 4, these two approaches lead to drastically different results. Both are intuitively reasonable, so which one should we choose?

A natural idea is to compare both accuracy estimates with the actual values of uncertainty. Interestingly, in several cases that we tried (see Section 5), the probabilistic result is too optimistic and the fuzzy result is too optimistic, the actual accuracy estimate is somewhere in between.

So, we need a new approach to come up with realistic estimates.

What we do in this paper. In this paper, we consider the general problem of combining accuracy. We show that, under reasonable assumptions, the corresponding combination can be described by a 1-parametric family.

In this family, two values of this parameter correspond to the naive probabilistic and naive fuzzy approaches.

In practice, for each domain, we recommend to find – and use – the most appropriate value of this parameter.

II. Preliminary Analysis

It is reasonable to consider the case when estimates are reasonably accurate. In some cases, the original expert estimates $\widetilde{x}_i$ are really ballpark estimates. This happens, e.g., when we ask experts to predict the distant future of technology or economy. In such cases, we understand that these estimates are not accurate at all, and, honestly, we do not worry how accurate the resulting estimate $\widetilde{y}$ is: of course, it is not very accurate, the difference from the actual value can be in orders of magnitude.

The problem of estimating the accuracy becomes important when the original estimates are reasonably accurate, i.e., when the differences Δx_i are reasonably small.

Linearization. We are interested in the difference

$$\Delta y = f\left(\widetilde{x}_1, \ldots, \widetilde{x}_n\right) - f(x_1, \ldots, x_n).$$

We do not know the actual values x_i, we only know that these values are close to the estimates $\widetilde{x}_i$, i.e., that $x_i = \widetilde{x}_i - \Delta x_i$, for some small values Δx_i. Substituting these expressions for x_i into the formula for Δy, we conclude that

$$\Delta y = f\left(\widetilde{x}_1, \ldots, \widetilde{x}_n\right) - f\left(\widetilde{x}_1 - \Delta x_1, \ldots, \widetilde{x}_n - \Delta x_n\right).$$

Since the values Δx_i are small, we can expand the above expression in Taylor series and keep only terms which are linear in Δx_i. As a result, we get the expression

$$\Delta y = \sum_{i=1}^{n} c_i \cdot \Delta x_i,$$

where $c_i \stackrel{\text{def}}{=} \dfrac{\partial f}{\partial x_i}$ are the partial derivatives of the function $f(x_1, \ldots, x_n)$ computed for $x_i = \widetilde{x}_i$. Since we know the function $f(x_1, \ldots, x_n)$, we can compute the corresponding derivatives c_i.

So, the quantity Δy is the sum of n terms $\delta x_i \stackrel{\text{def}}{=} c_i \cdot \Delta x_i$. Thus, to find the size of the difference Δ, let us first estimate the sizes of each of these terms.

Estimating the size of each terms δx_i. The approximation error can be positive or negative. In most cases, we have no reason to believe that positive values are more probable or less probable. In other words, there is no difference between our knowledge of possible values of the difference Δx_i and of the opposite-sign difference $-\Delta x_i$. This means, in particular, that the value $-\Delta x_i$ should have the same size Δ_i as the original difference Δx_i.

Another observation is that if we change the original measuring unit to a new one which is c times smaller, then all the numerical values get multiplied by c. For example, if we replace meters with centimeters, all the numerical values of length get multiplied by 100. In general, this means that:

- instead of the original value Δx_i, we now have the new value $c \cdot \Delta x_i$, and
- instead of the original accuracy estimate Δ_i, we now have a new estimate $c \cdot \Delta_i$.

Thus, if the approximation error Δx_i is of size Δ_i, then for $c_i > 0$, the quantity $c_i \cdot \Delta x_i$ is of size $c_i \cdot \Delta_i$. This takes care of multiplying Δx_i by a positive number. Multiplying the quantity Δx_i by a negative number $c_i < 0$ is equivalent to multiplying the quantity $-\Delta x_i$ by a positive number $|c_i|$. Since, as we have mentioned, for $-\Delta x_i$, the accuracy is the same Δ_i as for Δx_i, we thus conclude that for $c_i < 0$, the accuracy of $c_i \cdot \Delta x_i = |c_i| \cdot (-\Delta x_i)$ is characterized by the value $|c_i| \cdot \Delta_i$.

Combining the cases $c_i > 0$ and $c_i < 0$, we can make a general statement:

- if the approximation error Δx_i is of size Δ_i,
- then for each c_i, the quantity $\delta x_i = c_i \cdot \Delta x_i$ is of size

$$\delta_i \stackrel{\text{def}}{=} |c_i| \cdot \Delta_i.$$

Resulting problem. We have the sum $\Delta y = \sum_{i=1}^{n} \delta x_i$ of n terms δx_i each of which is of the size δ_i. What is the size of the sum?

Let us show how different reasonable approaches solve this problem.

III. Simple Probabilistic Approach

Analysis of the problem. There are many different reasons why an expert estimate $\widetilde{x}_i$ is, in general, different from the actual value x_i of the corresponding quantity. Thus, the difference $\Delta x_i = \widetilde{x}_i - x_i$ can be viewed as a sum of many different independent components corresponding to these different reasons.

It is known that the probability distribution of a sum of a large number of small independent random variables is, in general, close to Gaussian; this result is known as the *Central Limit Theorem* (see, e.g., [14]). Thus, it is reasonable to assume that each difference is normally distributed.

Each normal distribution is uniquely characterized by two parameters: mean m and standard deviation σ. As we have discussed in the previous section, it makes sense to assume that the difference Δx_i and its opposite $-\Delta x_i$ have the same accuracy characteristics. This means, in particular, that the mean value m_i of Δx_i should be the same as the mean value of $-\Delta x_i$. However, the mean value of $-\Delta x_i$ is equal to $-\mu_i$. Thus, $-m_i = m_i$, hence $m_i = 0$. Thus, the distribution of each difference Δx_i is characterized by only one parameter: its standard deviation σ_i.

So, in this case, we can take σ_i as the corresponding accuracy measure Δ_i. For the term $\delta x_i = c_i \cdot \Delta x_i$, the standard deviation is equal to $\delta_i = |c_i| \cdot \Delta_i$.

There is no reason to believe that there is any correlation between approximation errors Δx_i corresponding to different quantities x_i. Thus, it is reasonable to assume that the approximation errors Δx_i are independent – and thus, that the quantities $\delta x_i = c_i \cdot \Delta x_i$ are also independent. For the sum $\Delta y = \sum_{i=1}^{n} \delta x_i$ of several independent random variables, the variance δ^2 is equal to the sum of variances δ_i^2. So, we arrive at the following formula.

Resulting formula (see, e.g., [13]) **.**In the probabilistic approach, once we know the error estimates $\delta_i = |c_i| \cdot \Delta_i$ for the terms $\delta x_i = c_i \cdot \Delta x_i$, the error estimate Δ for the sum $\Delta y = \sum_{i=1}^{n} \delta x_i$ has the form

$$\Delta = \sqrt{\sum_{i=1}^{n} \delta_i^2} = \sqrt{\sum_{i=1}^{n} c_i^2 \cdot \Delta_i^2}. \qquad (1)$$

IV. Simple Fuzzy Approach

Analysis of the problem. In the fuzzy case, uncertainty is characterized by a membership function. Let $\mu_i(\Delta x_i)$ by the membership function that describes, for each theoretically possible value of the approximation error Δx_i, to what extent this particular value is actually possible.

We have decided to assume that the information about Δx_i is the same as information about $-\Delta x_i$, so the membership function should be event: $\mu_i(\Delta x_i) = \mu_i(|\Delta x_i|)$. It is also reasonable to assume that the larger the deviation, the less possible it is, i.e., that each function $\mu_i(z)$ is decreasing for $z \geq 0$.

We assume that each uncertainty is characterized by only one parameter Δ_i. Let $\mu_0(\Delta x_0)$ be a membership function corresponding to the value 1 of this parameter. If we now take a difference Δx_i and change the measuring unit to the one which is Δ_i larger, then all numerical values are divided by Δ_i, i.e., instead of each original value Δx_i, we get a new numerical value $\Delta x_i' = \dfrac{\Delta x_i}{\Delta_i}$. In particular, the new value of the accuracy parameter will be $\dfrac{\Delta_i}{\Delta_i} = 1$. Thus, the new variable $\Delta x_i'$ is characterized by the membership function $\mu_0(z)$ corresponding to the value 1 of the uncertainty parameter.

So, for each value Δx_i, the degree to which this value is possible is equal to

$$\mu_i(\Delta x_i) = \mu_0(\Delta x_i') = \mu_0\left(\frac{\Delta x_i}{\Delta_i}\right).$$

Similarly, we can conclude that for $\delta x_i = c_i \cdot \Delta x_i$, the membership function takes the form

$$\mu_i'(\delta x_i) = \mu_0\left(\frac{\delta x_i}{\delta_i}\right).$$

To form a membership function corresponding to the sum $\Delta y = \sum_{i=1}^{n} \delta x_i$ of the quantities δx_i, it is reasonable to use Zadeh's extension principle

$$\mu(\Delta y) = \max_{(\delta x_1, \ldots, \delta x_n):\, \sum_{i=1}^{n} \delta x_i = \Delta y} f_{\&}(\mu_1'(\delta x_1), \ldots, \mu_n'(\delta x_n)),$$

where $f_{\&}(a, b)$ is an appropriate "and"-operation (t-norm). In particular, for the simplest "and"-operation $f_{\&}(a, b) = \min(a, b)$, we get

$$\mu(\Delta y) = \max_{(\delta x_1, \ldots, \delta x_n):\, \sum_{i=1}^{n} \delta x_i = \Delta y} \min(\mu_1'(\delta x_1), \ldots, \mu_n'(\delta x_n)).$$

It is known that in the case when all the original membership functions have the same shape, the resulting function has exactly the same shape, namely, it has the form

$$\mu(\Delta y) = \mu_0\left(\frac{\Delta y}{\Delta}\right),$$

where $\Delta = \sum_{i=1}^{n} \delta_i$.

This formula can be easily confirmed if we take into account that, according to Zadeh's extension principle with $f_{\&}(a, b) = \min(a, b)$, each alpha-cut $\mathbf{y}(\alpha)$ for the sum is equal to the sum of alpha-cuts corresponding to each term. For each term, the alpha-cut is equal to $[-c \cdot \delta_i, c \cdot \delta_i]$, for some parameter c depending on α. Thus, the sum of these intervals has the form

$[-c\cdot\Delta, c\cdot\Delta]$, where $\Delta = \sum_{i=1}^{n} \delta_i$. So we arrive at the following formula.

Resulting formula. In the fuzzy approach, once we know the error estimates $\delta_i = |c_i| \cdot \Delta_i$ for the terms $\delta x_i = c_i \cdot \Delta x_i$, the error estimate Δ for the sum $\Delta y = \sum_{i=1}^{n} \delta x_i$ has the form

$$\Delta = \sum_{i=1}^{n} \delta_i = \sum_{i=1}^{n} |c_i| \cdot \Delta_i. \tag{2}$$

V. Resulting Challenge

Challenge. While both formulas (1) and (2) seem reasonable, they are different. They are very different: e.g., if all the value δ_i are the same, i.e., if $\delta_1 = \ldots = \delta_n$, then:

- in the probabilistic case, we get $\Delta = \sqrt{n} \cdot \delta_i$, while
- in the fuzzy case, we get $\Delta = n \cdot \delta_i$.

The difference is a factor of $\sqrt{n}$. When n is large – and we can have $n \approx 100$ – the difference is order of magnitude.

So which of the two approaches should we choose?

This is not about fuzzy vs. probabilistic. At first glance, this may look like a selection between probabilistic and fuzzy approaches, but it is not:

- the same formula (2) that we had in the fuzzy case can also be obtained in the probabilistic cases – when instead of normal distributions, we assumed a different type of distributions, Cauchy one; see, e.g., [7];
- similarly, the same formula (1) that we had in the probabilistic case can also be obtained in the fuzzy cases – when we consider another widely used "and"-operation $f_{\&}(a, b) = a\cdot b$ and frequently used Gaussian membership functions $\mu_i(\Delta x_i) = \exp\left(-\frac{(\Delta x_i)^2}{\Delta_i^2}\right)$.

This means that selecting a proper method is not a methodological question of which of the two approaches is better – fuzzy or probabilistic, it is a purely pragmatic question: how to combine accuracy estimates for Δx_i into an accuracy estimate for Δy.

We compared the two approaches on several examples. To decide which method works better, we compared these two approaches on several examples in which we could later determine the actual value of the quantity y; see, e.g., [2], [6]. Our conclusion was that that both methods are imperfect:

- the probabilistic formula (1) usually underestimated the uncertainty, while
- the fuzzy formula (2) usually overestimated the uncertainty.

So, what shall we do?

This seemingly negative result is actually in good accordance with Zadeh's ideas. Lotfi Zadeh always emphasized:

- that fuzzy logic is not a substitute for probabilities (or for any other uncertainty formalism),
- that an ideal way to deal with uncertainty is to realize that there are several different types of uncertainty and
- that, therefore, we need to combine different techniques.

This was the main idea behind the now well-accepted concept of *soft computing*.

So, instead of selecting one or another, let us try to combine the two approaches.

VI. How to Combine Accuracy Estimates: A General Approach

General idea. We want an operation $\delta_1 * \delta_2$ that combines the uncertainty characteristic $\delta_1 > 0$ and $\delta_2 > 0$ of two terms δx_1 and δx_2 into an uncertainty characteristic of their sum

$$\delta x_1 + \delta x_2.$$

Monotonicity. Intuitively, the sum cannot be more accurate than each of the values, so we must have

$$\delta_1 * \delta_2 \geq \delta_1 \text{ and } \delta_1 * \delta_2 \geq \delta_2.$$

Continuity. Small changes in δ_1 or in δ_2 should not lead to drastic changes in the result, so the operation should be *continuous*.

Commutativity. The sum does not depend on the order in which we add two quantities; thus, the value $\delta_1 * \delta_2$ should also not depend on the order. Thus, we should have

$$\delta_1 * \delta_2 = \delta_2 * \delta_1.$$

In mathematical terms, the desired operation should be *commutative*.

Associativity. If we want to combine three terms, we can do it in several different ways. For example:

- we can first combine δx_1 and δx_2, getting $\delta_1 * \delta_2$, and then combine the result with δx_3, resulting in

$$(\delta_1 * \delta_2) * \delta_3;$$

- alternatively, we can first combine δx_2 and δx_3, getting $\delta_2 * \delta_3$, and then combine the result with δx_1, resulting in

$$\delta_1 * (\delta_2 * \delta_3).$$

It is reasonable to require that the resulting estimate be the same in both case, i.e., that

$$(\delta_1 * \delta_2) * \delta_3 = \delta_1 * (\delta_2 * \delta_3).$$

In mathematical terms, the desired operation should be *associative*.

Scale-invariance. Finally, as we mentioned several times, the combination result should not change if we simply change the measuring unit to a new one which is c times smaller than the original one – after which all numerical values will be multiplied c. So, if we have $\delta = \delta_1 * \delta_2$, then, for every $c > 0$, we should have

$$c \cdot \delta = (c \cdot \delta_1) * (c \cdot \delta_2).$$

Now, we are ready for a formal definition.

Definition. *By a* combination operation*, we mean a continuous commutative associative binary operation $\delta_1 * \delta_2$ on the set of all non-negative numbers that satisfies the following two properties:*

- *for all δ_1 and δ_2, we have $\delta_1 * \delta_2 \geq \delta_1$, and*
- *for all δ_1, δ_2, and $c > 0$, if $\delta = \delta_1 * \delta_2$, then*

$$c \cdot \delta = (c \cdot \delta_1) * (c \cdot \delta_2).$$

Proposition. *Every combination operation has the form*

$$\delta_1 * \delta_2 = \max(\delta_1, \delta_2) \text{ or } \delta_1 * \delta_2 = (\delta_1^p + \delta_2^p)^{1/p}.$$

Proof. This proposition was, in fact, proven in [3].

Comment. The case $\delta_1 * \delta_2 = \max(\delta_1, \delta_2)$ is, in effect, the limit case of the formula $\delta_1 * \delta_2 = (\delta_1^p + \delta_2^p)^{1/p}$ when p tends to infinity. From this viewpoint, the $\max$-case can be viewed as a particular case of the general formula.

Resulting practical recommendation. Once we know the error estimates $\delta_i = |c_i| \cdot \Delta_i$ for the terms $\delta x_i = c_i \cdot \Delta x_i$, the error estimate Δ for the sum $\Delta y = \sum_{i=1}^{n} \delta x_i$ has the form

$$\Delta = \left(\sum_{i=1}^{n} \delta_i^p \right)^{1/p} = \left(\sum_{i=1}^{n} |c_i|^p \cdot \Delta_i^p \right)^{1/p}. \qquad (3)$$

Here:

- for $p = 2$, we get the probabilistic formula (1);
- for $p = 1$, we get the fuzzy formula (2).

In general, for each domain, we need to empirically select the value p.

Experimental assessment. To evaluate our method, we tested it on the example of seismic data. It turns out that, in this case, the most accurate estimates correspond to $p \approx 1.1$; this has been, in effect, shown in [2].

VII. Conclusions

In many practical situations:

- we know that the quantity y depends on the quantities $x_1, \ldots, x_n$, and, moreover, we know the exact form of this dependence dependence $y = f(x_1, \ldots, x_n)$;
- we know the approximate values $\widetilde{x}_1, \ldots, \widetilde{x}_n$ of the quantities x_i, and
- we know the accuracies $\Delta_1, \ldots, \Delta_n$ of these estimates.

Based on the estimates $\widetilde{x}_i$ for the values x_i, we can compute the estimate $\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ for y. What is the accuracy Δ of this estimate?

In this paper, we propose and justify the following formula for computing the desired estimate:

$$\Delta = \left(\sum_{i=1}^{n} |c_i|^p \cdot \Delta_i^p \right)^{1/p},$$

where $c_i \stackrel{\text{def}}{=} \dfrac{\partial f}{\partial x_i}$ are the partial derivatives of the function $f(x_1, \ldots, x_n)$ computed for $x_i = \widetilde{x}_i$.

The value p depends on the domain, it has to be experimentally determined for each domain, as the value for which the above formula is the closest to the actual accuracy of y.

For example, for the analysis of seismic data, the optimal value p is $p \approx 1.1$.

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References

[1] R. Belohlavek, J. W. Dauben, and G. J. Klir, *Fuzzy Logic and Mathematics: A Historical Perspective*, Oxford University Press, New York, 2017.

[2] D. I. Doser, K. D. Crain, M. R. Baker, V. Kreinovich, and M. C. Gerstenberger "Estimating uncertainties for geophysical tomography", *Reliable Computing*, 1998, Vol. 4, No. 3, pp. 241–268.

[3] A. Gholamy, J. Parra, V. Kreinovich, O. Fuentes, and E. Anthony, "How to best apply deep neural networks in geosciences: towards optimal 'averaging' in dropout training", In: J. Watada, S. C. Tan, P. Vasant, E. Padmanabhan, and L. C. Jain (eds.), *Smart Unconventional Modelling, Simulation and Optimization for Geosciences and Petroleum Engineering*, Springer Verlag, 2019, pp. 15–26.

[4] L. Jaulin, M. Kiefer, O. Didrit, and E. Walter, *Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control, and Robotics*, Springer, London, 2001.

[5] G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall, Upper Saddle River, New Jersey, 1995.

[6] V. Kreinovich, J. Beck, C. Ferregut, A. Sanchez, G. R. Keller, M. Averill, and S. A. Starks, "Monte-Carlo-type techniques for processing interval uncertainty, and their engineering applications", *Proceedings of the Workshop on Reliable Engineering Computing*, Savannah, Georgia, September 15–17, 2004, pp. 139–160.

[7] V. Kreinovich and S. Ferson, "A new Cauchy-based black-box technique for uncertainty in risk analysis", *Reliability Engineering and Systems Safety*, 2004, Vol. 85, No. 1–3, pp. 267–279.

[8] G. Mayer, *Interval Analysis and Automatic Result Verification*, de Gruyter, Berlin, 2017.

[9] J. M. Mendel, *Uncertain Rule-Based Fuzzy Systems: Introduction and New Directions*, Springer, Cham, Switzerland, 2017.

[10] R. E. Moore, R. B. Kearfott, and M. J. Cloud, *Introduction to Interval Analysis*, SIAM, Philadelphia, 2009.

[11] H. T. Nguyen, C. Walker, and E. A. Walker, *A First Course in Fuzzy Logic*, Chapman and Hall/CRC, Boca Raton, Florida, 2019.

[12] V. Novák, I. Perfilieva, and J. Močkoř, *Mathematical Principles of Fuzzy Logic*, Kluwer, Boston, Dordrecht, 1999.

[13] S. G. Rabinovich, *Measurement Errors and Uncertainties: Theory and Practice*, Springer, New York, 2005.

[14] D. J. Sheskin, *Handbook of Parametric and Nonparametric Statistical Procedures*, Chapman and Hall/CRC, Boca Raton, Florida, 2011.

[15] L. A. Zadeh, "Fuzzy sets", *Information and Control*, 1965, Vol. 8, pp. 338–353.