Can We Improve the Standard Algorithm of Interval Computation by Taking Almost Monotonicity into Account?

Martine Ceverio, Olga Kosheleva, and Vladik Kreinovich

Abstract In many practical situations, it is necessary to perform interval computations – i.e., to find the range of a given function $y = f(x_1,\ldots,x_n)$ on given intervals – e.g., when we want to find guaranteed bounds of a quantity that is computed based on measurements, and for these measurements, we only have upper bounds of the measurement error. The standard algorithm for interval computations first checks for monotonicity. However, when the function f is almost monotonic, this algorithm does not utilize this fact. In this paper, we show that such closeness-to-monotonicity can be efficiently utilized.

1 Formulation of the Problem

Need for interval computations. Most of the data comes from measurements. Measurements are never absolutely accurate, the measurement result $\widetilde{x}$ is, in general, different from from the actual (unknown) value of the corresponding quantity. In many cases, the only information that we have about the possible values of the measurement error $\Delta x \stackrel{\text{def}}{=} \widetilde{x} - x$ is the upper bounds Δ on its absolute value; see, e.g., [5].

Once we know this upper bound, then, based on the measurement result $\widetilde{x}$, the only thing that we can conclude about the actual value x is that this value belongs to the interval $\mathbf{x} = [\underline{x},\overline{x}] = [\widetilde{x}-\Delta,\widetilde{x}+\Delta]$.

In many practical situations, we are interested in a quantity y which itself is difficult (or even impossible) to measure but which depends, in a known way, on the values of several easier-to-measure quantities $x_1,\ldots,x_n$: $y = f(x_1,\ldots,x_n)$ [5].

Martine Ceberio and Vladik Kreinovich
Department of Computer Science, University of Texas at El Paso, El Paso, Texas 79968, USA
e-mail: mceberio@utep.edu, vladik@utep.edu

Olga Kosheleva
Department of Teacher Education, University of Texas at El Paso, El Paso, Texas 79968, USA
e-mail: olgak@utep.edu

In such situations, to estimate y, we measure the quantities x_i. When we only have upper bounds on all the measurement errors, then the only information that we have about each of the quantities x_i is that it belongs to the interval

$$\mathbf{x}_i = [\underline{x}_i, \overline{x}_i] = [\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i].$$

In this case, the set $\mathbf{y} = [\underline{y}, \overline{y}]$ of possible value of y is formed by the values $f(x_1, \ldots, x_n)$ corresponding to all possible combinations of values $x_i \in \mathbf{x}_i$:

$$\mathbf{y} = [\underline{y}, \overline{y}] = \{f(x_1, \ldots, x_n) : x_1 \in \mathbf{x}_1 \,\&\, \ldots \,\&\, x_n \in \mathbf{x}_n\}. \quad (1)$$

This range is also denoted by $f(\mathbf{x}_1, \ldots, \mathbf{x}_n)$.

Computing the values $\underline{y}$ and $\overline{y}$ based on the given algorithm $f(x_1, \ldots, x_n)$ and the given bounds $\underline{x}_i$ and $\overline{x}_i$ is known as *interval computations*; see, e.g., [1, 3, 4].

Need for computing enclosures. The above interval computations problem is, in general, NP-hard already for quadratic functions $f(x_1, \ldots, x_n)$; see, e.g., [2, 6]. This means that, unless P=NP (which most computer scientists believe to be false), no feasible algorithm is possible for solving all particular cases of the interval computation problem.

In many practically useful cases, we have efficient algorithms for interval computations. In other cases, all we can do is compute an *enclosure* $\mathbf{Y} \supseteq \mathbf{y}$ – and hope that this enclosure is close to the actual range $\mathbf{y}$.

The need for an enclosure comes from the fact that we often need to guarantee that the value y is within the given bounds $[y^-, y^+]$. If we produce an approximate solution $\mathbf{Y}$ and see that is it within the bounds, i.e., that $\mathbf{Y} \subseteq [y^-, y^+]$, then to guarantee that the actual value y is within the bound, we need to be sure that all possible values of y are in the set $\mathbf{Y}$, i.e., that $\mathbf{y} \subseteq \mathbf{Y}$ – i.e., that $\mathbf{Y}$ is indeed an enclosure.

Case of monotonic functions. There are cases when computing the range (1) is easy: e.g., it is easy when the function $f(x_1, \ldots, x_n)$ is monotonic in each of its variables. For example, if the function $f(x_1, \ldots, x_n)$ is increasing in each of its variables, then its range is equal to $[f(\underline{x}_1, \ldots, \underline{x}_n), f(\overline{x}_1, \ldots, \overline{x}_n)]$.

An important particular case of monotonicity is the case when the function $f(x_1, \ldots, x_n)$ is linear, i.e., when $f(x_1, \ldots, x_n) = a_0 + \sum\limits_{i=1}^{n} a_i \cdot x_i$. The range of the linear function is equal to $\mathbf{y} = [\widetilde{y} - \Delta, \widetilde{y} + \Delta]$, where $\widetilde{y} \stackrel{\text{def}}{=} f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ and $\Delta \stackrel{\text{def}}{=} \sum\limits_{i=1}^{n} |a_i| \cdot \Delta_i$. In particular, when $f(x_1, x_2) = x_1 + x_2$, we get

$$[\underline{x}_1, \overline{x}_1] + [\underline{x}_2, \overline{x}_2] = [\underline{x}_1 + \underline{x}_2, \overline{x}_1 + \overline{x}_2], \quad (2)$$

and when $f(x_1, x_2) = x_1 - x_2$, we get

$$[\underline{x}_1, \overline{x}_1] - [\underline{x}_2, \overline{x}_2] = [\underline{x}_1 - \overline{x}_2, \overline{x}_1 - \underline{x}_2]. \quad (3)$$

Interval arithmetic operations. Monotonicity covers two arithmetic operations: addition and subtraction.

For multiplication, the situation is somewhat more complex, but still:

- For each x_1, the function $f(x_1,x_2) = x_1 \cdot x_2$ is either increasing or decreasing with respect to x_2 and thus, attains its minimum and maximum at one of the two endpoints of the interval $[\underline{x}_2,\overline{x}_2]$.
- Similarly, for each x_2, this function is either increasing or decreasing with respect to x_1, so to find its extreme values, it is sufficient to consider only the two endpoints $\underline{x}_1$ and $\overline{x}_1$.

Thus, to find the minimum and maximum of the product, it is sufficient to only consider 4 combinations of endpoints:

$$[\underline{x}_1,\overline{x}_1]\cdot[\underline{x}_2,\overline{x}_2] =$$

$$[\min(\underline{x}_1\cdot\underline{x}_2,\underline{x}_1\cdot\overline{x}_2,\overline{x}_1\cdot\underline{x}_2,\overline{x}_1\cdot\overline{x}_2),\min(\underline{x}_1\cdot\underline{x}_2,\underline{x}_1\cdot\overline{x}_2,\overline{x}_1\cdot\underline{x}_2,\overline{x}_1\cdot\overline{x}_2)]. \quad (4)$$

The only remaining arithmetic operation is division x_1/x_2. In the computers, it is usually implemented as

$$x_1/x_2 = x_1 \cdot (1/x_2), \quad (5)$$

and the function $1/x_2$ is decreasing on any interval not containing 0, so

$$1/[\underline{x}_2,\overline{x}_2] = [1/\overline{x}_2, 1/\underline{x}_2] \text{ if } 0 \notin [\underline{x}_1,\overline{x}_2]. \quad (6)$$

Straightforward interval computations. In the computer, the only hardware supported operations are arithmetic operations, so any numerical algorithm is implemented as a sequence of such arithmetic operations; for example, computing $\sin(x)$ or $\exp(x)$ usually means computing the values of approximating polynomials (usually, the sum of the first few terms in the Taylor expansion).

Thus, one way to compute an enclosure is to replace each elementary arithmetic operation in the algorithm $f(x_1,\ldots,x_n)$ with the corresponding interval arithmetic operation (2)-(6).

The problem is that the resulting *straightforward* interval computations usually leads to a very wide enclosure.

Towards a standard algorithm for interval computations: first stage. To get a narrower enclosure, a natural idea is as follows.

First, we check whether the given problem allows *simple* interval computation – i.e., whether the function is monotonic. According to calculus, a function is increasing with respect to x_i if and only if the corresponding partial derivative is non-negative, i.e., if $\dfrac{\partial f}{\partial x_i} \ge 0$ on the whole box $\mathbf{x} \stackrel{\text{def}}{=} \mathbf{x}_1 \times \ldots \times \mathbf{x}_n$. To check this inequality, we can use, e.g., straightforward interval computations to find the enclosure $[\underline{D}_i,\overline{D}_i]$ for the range of this partial derivative.

- If the whole enclosure $[\underline{D}_i,\overline{D}_i]$ is non-negative, i.e., if $\underline{D}_i \geq 0$, this means that the function $f(x_1,\ldots,x_n)$ is increasing in x_i.
- If the whole enclosure $[\underline{D}_i,\overline{D}_i]$ is non-positive, i.e., if $\overline{D}_i \leq 0$, then the function $f(x_1,\ldots,x_n)$ is decreasing in x_i.

If the function $f(x_1,\ldots,x_n)$ is increasing or decreasing with respect to each variable, then we can immediately compute its range. If it is increasing or decreasing with respect to only some of the variables, then we can reduce the problem to computing ranges of functions of fewer variables. For example, if the function is increasing with respect to x_1, then:

- to estimate $\underline{y}$, it is sufficient to consider the smallest value $x_1 = \underline{x}_1$, i.e., to consider the minimum of the function $\underline{F}(x_2,\ldots,x_n) \stackrel{\text{def}}{=} f(\underline{x}_1,x_2,\ldots,x_n)$ of $n-1$ variables; and
- to estimate $\overline{y}$, it is sufficient to consider the largest value $x_1 = \overline{x}_1$, i.e., to consider the maximum of another function of $n-1$ variables:

$$\overline{F}(x_2,\ldots,x_n) \stackrel{\text{def}}{=} f(\overline{x}_1,x_2,\ldots,x_n).$$

Towards a standard algorithm for interval computations: second stage. If the original problem is not exactly simple, a natural next idea is to find a *nearby simple* problem and use its solution.

In interval computations, we approximate the original function by a linear one – this can be naturally done by using the Mean Value Theorem, according to which, for each combination of values $x_i = \widetilde{x}_i - \Delta x_i \in \mathbf{x}_i$, we have

$$f(x_1,\ldots,x_n) = f(\widetilde{x}_1 - \Delta x_1,\ldots,\widetilde{x}_n - \Delta x_n) =$$

$$f(\widetilde{x}_1,\ldots,\widetilde{x}_n) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\widetilde{x}_1 - \xi_1,\ldots,\widetilde{x}_n - \xi_n) \cdot \Delta x_i$$

for some $\xi_i \in [-\Delta_i,\Delta_i]$, leading to the following enclosure for the range $\mathbf{y} = f(\mathbf{x}_1,\ldots,\mathbf{x}_n)$:

$$\mathbf{y} \subseteq \widetilde{y} + \sum_{i=1}^{n} [\underline{D}_i,\overline{D}_i] \cdot [-\Delta_i.\Delta_i].$$

This formula is easy to compute – since the enclosures $[\underline{D}_i,\overline{D}_i]$ were estimated when we checked for monotonicity.

Towards a standard algorithm for interval computations: third stage. If the above method – of reducing a not-so-simple problem to a nearby simple one – does not lead to a sufficiently accurate enclosure, the next natural idea is to *divide the* original *problem into* several *simpler ones*. Usually:

- we divide one of the intervals $\mathbf{x}_i = [\underline{x}_i,\overline{x}_i]$ into two equal parts $\mathbf{x}'_i = [\underline{x}_i,\widetilde{x}_i]$ and $\mathbf{x}''_i = [\widetilde{x}_i,\overline{x}_i]$,
- we estimate the ranges $\mathbf{Y}'$ and $\mathbf{Y}''$ of the function $f(x_1,\ldots,x_n)$ on the corresponding sub-boxes $\mathbf{x}' = \mathbf{x}_1 \times \ldots \times \mathbf{x}_{i-1} \times \mathbf{x}'_i \times \mathbf{x}_{i+1} \times \ldots \times \mathbf{x}_n$ and

$$\mathbf{x}'' = \mathbf{x}_1 \times \ldots \times \mathbf{x}_{i-1} \times \mathbf{x}_i'' \times \mathbf{x}_{i+1} \times \ldots \times \mathbf{x}_n;$$

- we take the union of these ranges: $\mathbf{Y} = \mathbf{Y}' \cup \mathbf{Y}''$.

This method works since when we approximate a function by a linear expression (as in mean value form), we thus ignore quadratic (and higher order) terms; thus, the accuracy of the mean valued form is of order Δ_i^2. When we divide the interval into two halves, each with half of the original value Δ_i, the corresponding error component decreases 4 times, to $(\Delta_i/2)^2 = \Delta_i^2/4$.

If we want a more accurate enclosure, we can bisect further, etc.

Thus, we arrive at the following algorithm – which is used in most interval computations software packages.

The standard algorithm for interval computations: summary.

- first, we check if the function is monotonic with respect to (at least) some of the variables,
- then, we apply the mean value form,
- then, if needed, we bisect, and repeat the whole procedure for each sub-box.

A problem with the standard algorithm – and what we do in this paper. The problem with the standard algorithm is that:

- when it checks whether a given problem is already simple, it checks for the general monotonicity property;
- however, when this algorithm takes into account a nearby simple problem, it only considers linear cases – but not more general monotonic ones.

As a result:

- when the function is linear, we get the exact range,
- and when the function is almost linear – i.e., it is very close to a linear one – we get an almost correct range, that tends to the exact one when the difference from a linear function tends to 0.

On the other hand, for non-linear functions:

- when the function is monotonic, we get the exact range,
- but when the function is almost monotonic – i.e., very close to a monotonic one – we do not utilize this closeness and get, in general, a lousy estimate.

It is therefore desirable to take this “almost monotonicity” into account, i.e., to come up with a method that would lead to the exact range when the difference from monotonicity tends to 0.

This is what we do in this paper.

2 How to Take Almost Monotonicity into Account: 1-D Case

Discussion. Let us see which bounds on the endpoints $\underline{y}$ and $\overline{y}$ can be extracted from the condition of almost monotonicity, and let us check whether these bounds can be better than the bounds coming from the usual mean value form. To make this comparison, we need to describe which bounds come from the mean value form.

Which bounds come from the mean value form. In the 1-D case, the mean value form takes the form $f(\widetilde{x}_1)+[\underline{D}_1,\overline{D}_1]\cdot[-\Delta_1,\Delta_1]$. For the product of the two intervals, the upper endpoint is equal to

$$\max(\underline{D}_1\cdot\Delta_1,-\underline{D}_1\cdot\Delta_1,\overline{D}_1\cdot\Delta_1,-\overline{D}_1\cdot\Delta_1)=$$

$$\max(\max(\underline{D}_1\cdot\Delta_1,-\underline{D}_1\cdot\Delta_1),\max(\overline{D}_1\cdot\Delta_1,-\overline{D}_1\cdot\Delta_1)).$$

In general, $\max(a,-a)=|a|$, so the upper endpoint is equal to $\Delta_1\cdot\max(|\underline{D}_1|,|\overline{D}_1|)$. Thus, the upper bound for $f(x_1)$ takes the form

$$\overline{y}\le f(\widetilde{x}_1)+\Delta_1\cdot\max(|\underline{D}_1|,|\overline{D}_1|). \tag{7}$$

Similarly, the lower bound corresponding to the mean value form is of the type

$$\underline{y}\ge f(\widetilde{x}_1)-\Delta_1\cdot\max(|\underline{D}_1|,|\overline{D}_1|). \tag{8}$$

What does almost monotonicity mean in 1-D case. If the function $f(x_1)$ is increasing, then its range on the interval $[\underline{x}_1,\overline{x}_1]$ is equal to $[f(\underline{x}_1),f(\overline{x}_1)]$. The function is increasing if the range $[\underline{d}_1,\overline{d}_1]$ of its derivative contains only non-negative values, i.e., if $\underline{d}_1\ge 0$.

In practice, usually, instead of the actual range $[\underline{d}_1,\overline{d}_1]$, we only know the enclosure $[\underline{D}_1,\overline{D}_1]$ for this range. Thus, we can utilize the monotonicity property is $\underline{D}_1\ge 0$. In this terms, "almost monotonicity" means that the value $\underline{D}_1$ is negative but close to 0, i.e., has the form $\underline{D}_1=-\varepsilon$, for some small $\varepsilon>0$.

Similarly, since decreasing means that $\overline{D}_1\le 0$, almost decreasing means that $\overline{D}_1=\varepsilon>0$ for some small ε.

What bounds can we extract when the function $f(x_1)$ is almost increasing: lower bounds. For an almost increasing function, for each $x_1\in[\underline{x}_1,\overline{x}_1]$, we have

$$f(x_1)=f(\underline{x}_1)+\int_{\underline{x}_1}^{x_1} f'(x)\,dx.$$

Since $f'(x)\ge-\varepsilon$ for all $x_1\in[\underline{x}_1,\overline{x}_1]$, we can thus conclude that

$$f(x_1)\ge f(\underline{x}_1)-\varepsilon\cdot(x_1-\underline{x}_1).$$

Thus, for each x_1, we get $f(x_1)\ge\min\limits_{x_1}(f(\underline{x}_1)-\varepsilon\cdot(x_1-\underline{x}_1))$. The smallest value of the right-hand side is attained when x_1 is the largest, i.e., when $x_1=\overline{x}_1$. In this case, $x_1-\underline{x}_1=\overline{x}_1-\underline{x}_1=2\cdot\Delta_1$, thus, we conclude that

$$f(x_1) \geq f(\underline{x}_1) - 2 \cdot \varepsilon \cdot \Delta_1.$$

Thus, when $\underline{D}_1 < 0$, we have a lower bound $\underline{y} \geq f(\underline{x}_1) + 2 \cdot \underline{D}_1 \cdot \Delta_1$. When $\underline{D}_1 \geq 0$, the lower bound is exactly $f(\underline{x}_1)$. Thus, in general, we get the following lower bound:

$$\underline{y} \geq f(\underline{x}_1) + 2 \cdot \min(0, \underline{D}_1) \cdot \Delta_1. \tag{9}$$

We can easily see that if the function $f(x_1)$ is increasing, we get the exact lower bound $\underline{y} = f(\underline{x}_1)$.

What bounds can we extract when the function $f(x_1)$ is almost increasing: upper bounds. Similarly, from the fact that

$$f(\overline{x}_1) - f(x_1) = \int_{x_1}^{\overline{x}_1} f'(x)\, dx \geq -\varepsilon \cdot (\overline{x}_1 - x_1),$$

it follows that $f(x_1) \leq f(\overline{x}_1) + \varepsilon \cdot (\overline{x}_1 - x_1)$ and thus, that

$$f(x_1) \leq \max_{x_1} (f(\overline{x}_1) + \varepsilon \cdot (\overline{x}_1 - x_1)).$$

The largest value is attained when x_1 is the smallest, i.e., when $x_1 = \underline{x}_1$; in this case, $\overline{x}_1 - x_1 = 2 \cdot \Delta_1$, so we conclude that $f(x_1) \leq f(\overline{x}_1) + 2 \cdot \varepsilon \cdot \Delta_1$. So, in general, we get an upper bound

$$\overline{y} \leq f(\overline{x}_1) - 2 \cdot \min(0, \underline{D}_1) \cdot \Delta_1. \tag{10}$$

If the function $f(x_1)$ is increasing, we get the exact upper bound $\overline{y} = f(\overline{x}_1)$.

What bounds can we extract when the function $f(x_1)$ is almost decreasing: upper bounds. For an almost decreasing function, for each $x_1 \in [\underline{x}_1, \overline{x}_1]$, from

$$f(x_1) = f(\underline{x}_1) + \int_{\underline{x}_1}^{x_1} f'(x)\, dx,$$

by using the fact that $f'(x) \leq \varepsilon$, we can thus conclude that

$$f(x_1) \leq f(\underline{x}_1) + \varepsilon \cdot (x_1 - \underline{x}_1).$$

Thus, for each x_1, we get

$$f(x_1) \geq \max_{x_1} (f(\underline{x}_1) + \varepsilon \cdot (x_1 - \underline{x}_1)).$$

The largest value of the right-hand side is attained when x_1 is the largest, i.e., when $x_1 = \overline{x}_1$. In this case, $x_1 - \underline{x}_1 = \overline{x}_1 - \underline{x}_1 = 2 \cdot \Delta_1$, thus, we conclude that

$$f(x_1) \leq f(\underline{x}_1) + 2 \cdot \varepsilon \cdot \Delta_1.$$

Thus, when $\overline{D}_1 > 0$, we have an upper bound $\overline{y} \leq f(\underline{x}_1) + 2 \cdot \overline{D}_1 \cdot \Delta_1$. When $\overline{D}_1 \leq 0$, the upper bound is exactly $f(\underline{x}_1)$. Thus, in general, we get the following upper bound:

$$\overline{y} \geq f(\underline{x}_1) + 2 \cdot \max(0, \overline{D}_1) \cdot \Delta_1. \tag{11}$$

If the function $f(x_1)$ is decreasing, we get the exact upper bound $\overline{y} = f(\underline{x}_1)$.

What bounds can we extract when the function $f(x_1)$ is almost decreasing: lower bounds. Similarly, from the fact that

$$f(\overline{x}_1) - f(x_1) = \int_{x_1}^{\overline{x}_1} f'(x)\,dx \geq -\varepsilon \cdot (\overline{x}_1 - x_1),$$

it follows that $f(x_1) \geq f(\overline{x}_1) - \varepsilon \cdot (\overline{x}_1 - x_1)$ and thus, that

$$f(x_1) \geq \min_{x_1}(f(\overline{x}_1) - \varepsilon \cdot (\overline{x}_1 - x_1)).$$

The smallest value is attained when x_1 is the smallest, i.e., when $x_1 = \underline{x}_1$; in this case, $\overline{x}_1 - x_1 = 2 \cdot \Delta_1$, so we conclude that $f(x_1) \geq f(\overline{x}_1) - 2 \cdot \varepsilon \cdot \Delta_1$. So, in general, we get a lower bound

$$\underline{y} \leq f(\overline{x}_1) - 2 \cdot \max(0, \overline{D}_1) \cdot \Delta_1. \tag{12}$$

If the function $f(x_1)$ is decreasing, we get the exact lower bound $\underline{y} = f(\overline{x}_1)$.

Summarizing. From the mean value form, we extract the bounds

$$\underline{y} \geq f(\widetilde{x}_1) - \Delta_1 \cdot \max(|\underline{D}_1|, |\overline{D}_1|) \text{ and } \overline{y} \leq f(\widetilde{x}_1) + \Delta_1 \cdot \max(|\underline{D}_1|, |\overline{D}_1|).$$

From the almost-increasing idea, we can conclude that

$$\underline{y} \geq f(\underline{x}_1) + 2 \cdot \min(0, \underline{D}_1) \cdot \Delta_1 \text{ and } \overline{y} \leq f(\overline{x}_1) - 2 \cdot \min(0, \underline{D}_1) \cdot \Delta_1.$$

From the almost-decreasing idea, we can conclude that

$$\underline{y} \leq f(\overline{x}_1) - 2 \cdot \max(0, \overline{D}_1) \cdot \Delta_1 \text{ and } \overline{y} \geq f(\underline{x}_1) + 2 \cdot \max(0, \overline{D}_1) \cdot \Delta_1.$$

Resulting idea. In the non-monotonic case:

- instead of only computing the bounds $\underline{Y}$ and $\overline{Y}$ corresponding to the mean value form,
- why not also (or instead) compute the bounds corresponding to almost-monotonicity (at least some of them), and then take the largest of the lower bounds $\underline{Y}$ and the smallest of the upper bounds $\overline{Y}$?

To decide when to use the new bounds, let us analyze, on a simple example, when the next bounds are better.

What is the simplest case on which we can compare the new method with the existing ones. Linear functions are monotonic – and thus, for linear function, we do not need neither the mean value bounds, not any new bounds, we can easily compute the actual range.

Thus, to compare the two methods, we need to consider nonlinear functions. Let us consider the simplest nonlinear functions: the quadratic functions $f(x_1) = a_0 \cdot x_1^2 + a_1 \cdot x_1 + a_2$. Without losing generality, let us consider the case when $a_0 > 0$.

When we re-scale x_1 to $x_1' \stackrel{\text{def}}{=} \sqrt{a_0} \cdot x_1$, the interval changes, but, as one can see, the ranges produced by all three methods do not change. Thus, to compare the methods, it is sufficient to consider the case when $a_0 = 1$.

We can always represent the resulting expression $x_1^2 + a_1 \cdot x_1 + a_2$ as

$$(x_1 + a_1/2)^2 + \text{const.}$$

Replacing x_1 with the new variable $x_1' = x_1 + a_1/2$ also does not change the bounds, so we can safely assume that $a_1 = 0$, and thus, $f(x_1) = x_1^2 + a_2$.

If we subtract a constant from all the values of the function $f(x_1)$, then all the bounds will decrease by the same constant – and thus, this change will not affect which bound is smaller or larger (and hence, will not affect which method is better). So, for the purpose of comparing the estimates, we can subtract a_2 from all the values of the function $f(x_1)$ and thus, consider the function $f(x_1) = x_1^2$.

We consider the case when the function is not monotonic on the interval $[\underline{x}_1, \overline{x}_1]$, so we must have $\underline{x}_1 < 0 < \overline{x}_1$ – since on intervals not containing 0 the function $f(x_1) = x_1^2$ is clearly monotonic.

Without losing generality, let us assume that $|\underline{x}_1| \le \overline{x}_1$: if this is not true, we can switch from x_1 to $-x_1$ and get the above inequality – and this switch also does not affect the relative comparison of the above estimates.

Comparing the estimates on the simplest case: general discussion. Let us consider this example of a function $f(x_1) = x_1^2$ on an interval $[\underline{x}_1, \overline{x}_1]$ for which $\underline{x}_1 < 0 < \overline{x}_1$ and $|\underline{x}_1| \le \overline{x}_1$. In this case, the size $\overline{x}_1$ of the increasing part is larger than (or equal to) the size $|\underline{x}_1|$ of the decreasing part. In this sense, the function is more on the increasing side, so it makes sense to compare the mean value estimate only with the almost increasing case.

Here, $\widetilde{x}_1 = \dfrac{\underline{x}_1 + \overline{x}_1}{2} = \dfrac{\overline{x}_1 - |\underline{x}_1|}{2}$ and $\Delta_1 = \dfrac{\overline{x}_1 - \underline{x}_1}{2} = \dfrac{\overline{x}_1 + |\underline{x}_1|}{2}$. In this example, $f'(x_1) = 2x_1$, so $\underline{D}_1 = 2 \cdot \underline{x}_1 = -2|\underline{x}_1|$ and $\overline{D}_1 = 2 \cdot \overline{x}_1$. Here,

$$[\underline{D}_1, \overline{D}_1] \cdot [-\Delta_1, \Delta_1] = [-\overline{D}_1 \cdot \Delta_1, \overline{D}_1 \cdot \Delta_1].$$

Mean value bounds for the simplest case. The mean value estimate has the following form

$$\underline{Y} = \left(\frac{\overline{x}_1 - |\underline{x}_1|}{2} \right)^2 - \overline{D}_1 \cdot \Delta_1 = \left(\frac{\overline{x}_1 - |\underline{x}_1|}{2} \right)^2 - 2 \cdot \overline{x}_1 \cdot \frac{\overline{x}_1 + |\underline{x}_1|}{2} =$$

$$\frac{1}{4} \cdot (\overline{x}_1)^2 - \frac{1}{2} \cdot \overline{x}_1 \cdot |\underline{x}_1| + \frac{1}{4} \cdot (\underline{x}_1)^2 - (\overline{x}_1)^2 - \overline{x}_1 \cdot |\underline{x}_1| =$$

$$-\frac{3}{4}\cdot(\overline{x}_1)^2-\frac{3}{2}\cdot\overline{x}_1\cdot|\underline{x}_1|+\frac{1}{4}\cdot(\underline{x}_1)^2; \tag{13}$$

$$\overline{Y}=\left(\frac{\overline{x}_1-|\underline{x}_1|}{2}\right)^2+\overline{D}_1\cdot\Delta_1=\left(\frac{\overline{x}_1-|\underline{x}_1|}{2}\right)^2+2\cdot\overline{x}_1\cdot\frac{\overline{x}_1+|\underline{x}_1|}{2}=$$

$$\frac{1}{4}\cdot(\overline{x}_1)^2-\frac{1}{2}\cdot\overline{x}_1\cdot|\underline{x}_1|+\frac{1}{4}\cdot(\underline{x}_1)^2+(\overline{x}_1)^2+\overline{x}_1\cdot|\underline{x}_1|=$$

$$\frac{5}{4}\cdot(\overline{x}_1)^2+\frac{1}{2}\cdot\overline{x}_1\cdot|\underline{x}_1|+\frac{1}{4}\cdot(\underline{x}_1)^2. \tag{14}$$

Almost-increasing bounds for the simplest case. The almost-increasing estimates take the form

$$\underline{Y}=(\underline{x}_1)^2-2\cdot|\underline{x}_1|\cdot(\overline{x}_1+|\underline{x}_1|)=-2\cdot|\underline{x}_1|\cdot\overline{x}_1-(\underline{x}_1)^2; \tag{15}$$

$$\overline{Y}=(\overline{x}_1)^2+2\cdot|\underline{x}_1|\cdot(\overline{x}_1+|\underline{x}_1|)=(\overline{x}_1)^2+2\cdot|\underline{x}_1|\cdot\overline{x}_1+2\cdot(\underline{x}_1)^2. \tag{16}$$

When does the new method lead to a better lower bound? The lower bound produced by the new technique is better than the lower bound coming from the mean value form if

$$-\frac{3}{4}\cdot(\overline{x}_1)^2-\frac{3}{2}\cdot\overline{x}_1\cdot|\underline{x}_1|+\frac{1}{4}\cdot(\underline{x}_1)^2<-2\cdot|\underline{x}_1|\cdot\overline{x}_1-(\underline{x}_1)^2,$$

i.e., equivalently, if

$$\frac{5}{4}\cdot(\underline{x}_1)^2+\frac{1}{2}\cdot|\underline{x}_1|\cdot\overline{x}_1-\frac{3}{4}\cdot(\overline{x}_1)^2<0.$$

Multiplying both sides of this inequality by 4 and dividing by $(\overline{x}_1)^2$, we conclude that that $5z^2+2z-3<0$, where we denoted $z\stackrel{\text{def}}{=}\frac{|\underline{x}_1|}{\overline{x}_1}$. This is equivalent to $z<0.6$ – so it is better in most cases!

When does the new method lead to a better upper bound? The upper bound produced by the new technique is better then the upper bound coming from the mean value if

$$(\overline{x}_1)^2+2\cdot|\underline{x}_1|\cdot\overline{x}_1+2\cdot(\underline{x}_1)^2<\frac{5}{4}\cdot(\overline{x}_1)^2+\frac{1}{2}\cdot\overline{x}_1\cdot|\underline{x}_1|+\frac{1}{4}\cdot(\underline{x}_1)^2,$$

i.e., equivalently, that

$$\frac{7}{4}\cdot(\underline{x}_1)^2+\frac{3}{2}\cdot\overline{x}_1\cdot|\underline{x}_1|-\frac{1}{4}\cdot(\overline{x}_1)^2.$$

Multiplying both sides of this inequality by 4 and dividing by $(\overline{x}_1)^2$, we conclude that that $7z^2+6z-1<0$, i.e., that $z<1/7$. Thus, the upper bound is better only if $|\underline{x}_1| < \frac{1}{7}\cdot\overline{x}_1$ – i.e., only when the function is indeed almost increasing.

Towards a general conclusion. In the above simplest case, D_1 is simply proportional to x_1, with a positive coefficient (equal to 2). So, the comparison between the values $\underline{x}_1$ and $\overline{x}_1$ are equivalent to a similar comparison between $\underline{D}_1$ and $\overline{D}_1$. For example, the inequality $|\underline{x}_1| < 0.6\cdot\overline{x}_1$ is equivalent to $|\underline{D}_1| < 0.6\cdot\overline{D}_1$. Such inequalities can be easily formulated in the general case. Thus, we arrive at the following recommendations.

Comparison: resulting recommendation. When $\underline{D}_1 \ge 0$ or $\overline{D}_1 \le 0$, the function is monotonic – so, computing its range is easy. The difficulties come when $\underline{D}_1 < 0 < \overline{D}_1$. Let us consider the case when $|\underline{D}_1| \le \overline{D}_1$. Then, if we do not want to waste time on computing more than one pair of bounds:

- if $|\underline{D}_1| < \frac{1}{7}\cdot\overline{D}_1$, then we should only compute almost-increasing bounds;
- if $\frac{1}{7}\cdot\overline{D}_1 < |\underline{x}_1| \le 0.6\cdot\overline{D}_1$, then we should use the almost-increasing estimate to compute the lower bound and the mean value estimate for the upper bound;
- finally, if $0.6\cdot\overline{D}_1 \le |\underline{D}_1|$, we should only compute the mean value estimates.

Similarly, if $|\underline{D}_1| > \overline{D}_1$, then:

- if $\overline{D}_1 < \frac{1}{7}\cdot|\underline{D}_1|$, then we should only compute almost-decreasing bounds;
- if $\frac{1}{7}\cdot|\underline{D}_1| < \overline{D}_1 \le 0.6\cdot\overline{D}_1$, then we should use the mean value estimate for the lower bound and the almost-decreasing estimate to compute the upper bound;
- finally, if $0.6\cdot|\underline{D}_1| \le \overline{D}_1$, we should only compute the mean value estimates.

Comment. Of course, these recommendations are approximate, a future practical experience will hopefully provide better recommendations.

3 Multi-D Case

Almost increasing case: formulas. In the general multi-D case, we can similarly conclude that when $\underline{D}_i < 0 < \overline{D}_i$ and $|\underline{D}_i| \le \overline{D}_i$, then $\underline{y} \ge \underline{y}_{\underline{i}} - 2\cdot|\underline{D}_i|\cdot\Delta_i$, where $\underline{y}_{\underline{i}}$ denotes the smallest possible value of the following auxiliary function of $n-1$ variables

$$f_{\underline{i}}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) \stackrel{\text{def}}{=} f(x_1,\ldots,x_{i-1},\underline{x}_i,x_{i+1},\ldots,x_n).$$

Similarly, we have $\overline{y} \le \overline{y}_{\overline{i}} + 2\cdot|\underline{D}_i|\cdot\Delta_i$, where $\overline{y}_{\overline{i}}$ is the maximum of the function

$$f_{\overline{i}}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) \stackrel{\text{def}}{=} f(x_1,\ldots,x_{i-1},\overline{x}_i,x_{i+1},\ldots,x_n).$$

Almost increasing case: recommendation. Similar to the 1-D case, we can conclude that, if $|\underline{D}_i| < \frac{1}{7}\cdot\overline{D}_i$, then, instead of using the mean value bounds, we should reduce our original n-variable interval computations problem to two $(n-1)$-variable ones, for $f_{\underline{i}}$ and $f_{\overline{i}}$. Once we solve these fewer-variable problems and find the bounds $\underline{Y}_{\underline{i}} \le \underline{y}_{\underline{i}}$ and $\overline{Y}_{\overline{i}} \ge \overline{y}_{\overline{i}}$, we can then use the above formulas to produce the bounds for the original problem:

$$\underline{Y} = \underline{Y}_{\underline{i}} - 2\cdot|\underline{D}_i|\cdot\Delta_i \text{ and } \overline{Y} = \overline{Y}_{\overline{i}} + 2\cdot|\underline{D}_i|\cdot\Delta_i.$$

When $\frac{1}{7}\cdot\overline{D}_i \le |\underline{D}_i| < 0.6\cdot\overline{D}_i$, then we should use the above reduction to compute the lower bound $\underline{Y}$, and the mean value technique for computing the upper bound $\overline{Y}$.

When $0.6\cdot\overline{D}_i \le |\underline{D}_i| \le \overline{D}_i$, we should use mean value estimates for both bounds.

Almost decreasing case: recommendation. Similarly, if $\overline{D}_i < \frac{1}{7}\cdot|\underline{D}_i|$, then we should reduce our original n-variable interval computations problem to two $(n-1)$-variable ones. Once we solve these fewer-variable problems and find the bounds $\underline{Y}_{\overline{i}} \le \underline{y}_{\overline{i}}$ and $\overline{Y}_{\underline{i}} \ge \overline{y}_{\underline{i}}$, we can then produce the bounds for the original problem:

$$\underline{Y} = \underline{Y}_{\overline{i}} - 2\cdot\overline{D}_i\cdot\Delta_i \text{ and } \overline{Y} = \overline{Y}_{\underline{i}} + 2\cdot\overline{D}_i\cdot\Delta_i.$$

When $\frac{1}{7}\cdot|\underline{D}_i| \le \overline{D}_i < 0.6\cdot|\underline{D}_i|$, then we should use the above reduction to compute the lower bound $\underline{Y}$, and the mean value technique for computing the upper bound $\overline{Y}$.

When $0.6\cdot\overline{D}_i \le |\underline{D}_i| \le \overline{D}_i$, we should use mean value estimates for both bounds.

Acknowledgements This work was supported in part by the US National Science Foundation grant HRD-1242122 (Cyber-ShARE Center of Excellence).

References

1. L. Jaulin, M. Kiefer, O. Didrit, and E. Walter, *Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control, and Robotics*, Springer, London, 2001.
2. V. Kreinovich, A. Lakeyev, J. Rohn, and P. Kahl, *Computational Complexity and Feasibility of Data Processing and Interval Computations*, Kluwer, Dordrecht, 1998.
3. G. Mayer, *Interval Analysis and Automatic Result Verification*, de Gruyter, Berlin, 2017.
4. R. E. Moore, R. B. Kearfott, and M. J. Cloud, *Introduction to Interval Analysis*, SIAM, Philadelphia, 2009.
5. S. G. Rabinovich, *Measurement Errors and Uncertainties: Theory and Practice*, Springer, New York, 2005.
6. S. A. Vavasis, *Nonlinear Optimization: Complexity Issues*, Oxford University Press, New York, 1991.