How to Define “and”- and “or”-Operations for Intuitionistic and Picture Fuzzy Sets

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Abstract

The traditional fuzzy logic does not distinguish between the cases when we know nothing about a statement S and the cases when we have equally convincing arguments for S and for its negation $\neg S$: in both cases, we assign the degree 0.5 to such a statement S. This distinction is provided by *intuitionistic fuzzy logic*, when to describe our degree of confidence in a statement S, we use two numbers a_+ and a_- that characterize our degree of confidence in S and in $\neg S$. An even more detailed distinction is provided by *picture fuzzy logic*, in which we differentiate between cases when we are still trying to understand the truth value and cases when we are no longer interested. The question is how to extend “and”- and “or”-operations to these more general logics. In this paper, we provide a general idea for such extension, an idea that explain several extensions that have been proposed and successfully used.

1 Formulation of the Problem

Traditional fuzzy logic: brief reminder. In the traditional fuzzy logic (see, e.g., [2, 4, 5, 6, 7, 8]), we describe our degree of confidence in a statement A by a number $a \in [0, 1]$, so that:

- 0 means no confidence,
- 1 means full confidence, and
- intermediate values describe intermediate degrees of confidence.

In practice, there is often a need to estimate the degree of confidence in composite statements like $A \,\&\, B$ and $A \vee B$. There are many such statements, so it is not feasible to ask the experts about all of them. Instead, we must estimate our degree of confidence in $A \,\&\, B$ and $A \vee B$ based on our degrees of confidence a and b in the statements A and B. These estimates $f_\&(a,b)$ and $f_\vee(a,b)$ are known as *"and"-operation* (*t-norm*) and *"or"-operation* (*t-conorm.*

- The most widely used operations are $f_\&(a,b) = \min(a,b)$ and $f_\vee(a,b) = \max(a,b)$.
- Another very popular choice is $f_\&(a,b) = a \cdot b$ and $f_\vee(a,b) = a + b - a \cdot b$.

Need for intuitionistic fuzzy logic. In the traditional fuzzy logic, we do not distinguish between:

- the cases when we know nothing about A and
- the cases when we have equally strong arguments for and against A.

In both types of cases, we assign $a = 0.5$. To make this distinction, we can use two degrees:

- the degree a_+ of confidence in A and
- the degree of confidence a_- in $\neg A$,

the degrees for which $a_+ + a_- \le 1$. In this *intuitionistic fuzzy* approach (see, e.g., [1]):

- in the first case, we have $a_+ = a_- = 0$, and
- in the second case, we have $a_+ = a_- = 0.5$.

To define "and"- and "or"-operations to intuionistic fuzzy sets, we can find $f(a,b)$ for which $a_+ + a_- \le 1$ and $b_+ + b_- \le 1$ always imply $f_\&(a_+, b_+) + f(a_-, b_-) \le 1$.

Picture fuzzy sets. If $a_+ + a_- < 1$, this means that we do not have enough evidence for or against A.

- This may mean that we are still trying.
- This may mean that we are not interested in A at all.
- This may also mean that we are interested to some degree a_0, for which $a_+ + a_- + a_0 \le 1$.

How do we define "and"- and "or"-operations on such *picture sets* (see, e.g., [3])? Again, a similar idea is to find a function $f_0(a,b)$ for which $a_+ + a_- + a_0 \le 1$ and $b_+ + b_- + b_0 \le 1$ always imply $f_\&(a_+, b_+) + f(a_-, b_-) + f_0(a_0, b_0) \le 1$.

Problem. What operations should we define? In this paper, we use the above idea to describe possible "and"- and "or"-operations for intuitionstic and picture sets. These turn out to be the same operations that have been shown to be practically successful – but now we have a theoretical explanation for these heuristic operations.

2 Case of Intuitionsitic Fuzzy Sets

Definition 2.1. *Let $f_{\&}(a,b)$ be an "and"-operation. We say that $f(a,b)$ is an* intuitionistic operation corresponding to $f_{\&}(a,b)$ *if the following two properties are satisfied:*

- *first, if $a_+ + a_- \le 1$ and $b_+ + b_- \le 1$, then*

$$f_{\&}(a_+,b_+) + f(a_-,b_-) \le 1;$$

- *second, for each pair (a_-,b_-), there exist values a_+ and b_+ for which $a_+ + a_- \le 1$, $b_+ + b_- \le 1$, and*

$$f_{\&}(a_+,b_+) + f(a_-,b_-) = 1.$$

Proposition 2.1. *For each "and"-operation $f_{\&}(a,b)$, the corresponding intuitionistic operation has the form*

$$f(a,b) = 1 - f_{\&}(1-a,1-b).$$

Corollary 2.1 *For each "and"-operation (t-norm) $f_{\&}(a,b)$, the corresponding intuitionistic operation is an "or"-operation (t-conorm).*

Corollary 2.2. *For $f_{\&}(a,b) = \min(a,b)$, the corresponding intuitionistic operation is $f(a,b) = \max(a,b)$.*

Corollary 2.3. *For $f_{\&}(a,b) = a \cdot b$, the corresponding intuitistic operation is $f(a,b) = a + b - a \cdot b$.*

Definition 2.2. *Let $f_{\vee}(a,b)$ be an "or"-operation. We say that $f(a,b)$ is an* intuitionsitic operation corresponding to $f_{\vee}(a,b)$ *if the following two properties are satisfied:*

- *first, if $a_+ + a_- \le 1$ and $b_+ + b_0 \le 1$, then*

$$f_{\vee}(a_+,b_+) + f(a_-,b_-) \le 1;$$

- *second, for each pair (a_0,b_0), there exists values a_+, b_+, a_-, and b_- for which $a_+ + a_- \le 1$, $b_+ + b_0 \le 1$, and*

$$f_{\vee}(a_+,b_+) + f(a_-,b_-) = 1.$$

Proposition 2.2. *For each "or"-operation $f_{\vee}(a,b)$, the corresponding intuitionistic operation has the form*

$$f(a,b) = 1 - f_{\vee}(1-a,1-b).$$

Corollary 2.4. *For each "or"-operation (t-conorm) $f_\vee(a,b)$, the corresponding intuitionistic operation is an "and"-operation (t-norm).*

Corollary 2.5. *For $f_\vee(a,b)=\max(a,b)$, the corresponding intuitionistic operation is $f(a,b)=\min(a,b)$.*

Corollary 2.6. *For $f_\vee(a,b)=a+b-a\cdot b$, the corresponding intuitionistic operation is $f(a,b)=a\cdot b$.*

3 Case of Picture Fuzzy Sets

Comment. Based on the results of Section 2, we conclude that in intuitionistic fuzzy logic:

- if we start with an "and"-operation, we end up with an "or"-operation, and
- vice versa, if we start with an "or"-operation, we end up with an "and"-operation.

For each "and"-operation, we have a very specific "or"-operation and vice versa. However, for the purpose of generality, it makes sense to also consider the case when instead of a related pair of "and"- and "or"-operations, we have a general pair of such operations, with the only condition that whenever $a_+ + a_- \le 1$ and $b_+ + b_- \le 1$, then we have

$$f_\&(a_+,b_+)+f_\vee(a_-,b_-)\le 1.$$

The dual "and"- and "or"-operations – as described by Propositions 2.1 and 2.2 – always satisfy this condition.

Definition 3.1. *We say that an "and"-operation $f_\&(a,b)$ and an "or"-operation are* compatible *if whenever $a_+ + a_- \le 1$ and $b_+ + b_- \le 1$, we have*

$$f_\&(a_+,b_+)+f_\vee(a_-,b_-)\le 1.$$

Definition 3.2. *Let $f_\&(a,b)$ be an "and"-operation, and let $f_\vee(a,b)$ be a compatible "or"-operation. We say that $f_0(a,b)$ is a* picture "and"-operation *corresponding to $f_\&(a,b)$ and $f_\vee(a,b)$ if the following two properties are satisfied:*

- *first, if $a_+ + a_0 + a_- \le 1$ and $b_+ + b_- + b_0 \le 1$, then*

$$f_\&(a_+,b_+)+f_0(a_0,b_0)+f_\vee(a_-,b_-)\le 1;$$

- *second, for each pair (a_0,b_0), there exists values a_+, b_+, a_-, and b_- for which $a_+ + a_0 + a_- \le 1$, $b_+ + b_- + b_0 \le 1$, and*

$$f_\&(a_+,b_+)+f_0(a_0,b_0)+f_\vee(a_-,b_-)=1.$$

Proposition 3.1. *For each "and"-operation $f_\&(a,b)$ and each compatible "or"-operation $f_\vee(a,b)$, the corresponding picture "and"-operation has the form*

$$f_0(a,b) = 1 - \max_{a_+\le 1-a, b_+\le 1-b} (f_\&(a_+,b_+) + f_\vee(1-a-a_+, 1-b-b_+)).$$

Proposition 3.2. *For $f_\&(a,b) = \min(a,b)$ and $f_\vee(a,b) = \max(a,b)$, the corresponding picture "and"-operation is $f_0(a,b) = \min(a,b)$.*

Proposition 3.3. *For $f_\&(a,b) = a\cdot b$ and $f_\vee(a,b) = a+b-a\cdot b$, the corresponding picture "and"-operation is $f_0(a,b) = a\cdot b$.*

Definition 3.3. *Let $f_\vee(a,b)$ be an "or"-operation, and let $f_\&(a,b)$ be a compatible "and"-operation. We say that $f_0(a,b)$ is a* picture "or"-operation *corresponding to $f_\vee(a,b)$ and $f_\&(a,b)$ if the following two properties are satisfied:*

- *first, if $a_+ + a_0 + a_- \le 1$ and $b_+ + b_- + b_0 \le 1$, then*

$$f_\vee(a_+,b_+) + f_0(a_0,b_0) + f_\&(a_-,b_-) \le 1;$$

- *second, for each pair (a_0,b_0), there exists values a_+, b_+, a_-, and b_- for which $a_+ + a_0 + a_- \le 1$, $b_+ + b_- + b_0 \le 1$, and*

$$f_\vee(a_+,b_+) + f_0(a_0,b_0) + f_\&(a_-,b_-) = 1.$$

Proposition 3.4. *For each "and"-operation $f_\&(a,b)$ and each compatible "or"-operation $f_\vee(a,b)$, the corresponding picture "or"-operation has the form*

$$f_0(a,b) = 1 - \max_{a_+\le 1-a, b_+\le 1-b} (f_\vee(a_+,b_+) + f_\&(1-a-a_+, 1-b-b_+)).$$

The Propositions 3.1 and 3.4 leads to a somewhat unexpected conclusion:

Corollary 3.1. *For each pair of compatible "and"- and "or"-operations, the corresponding picture "or"-operation is the same as the corresponding picture "and"-operation.*

4 Possible Extensions: Idea

Fuzzy logic → intuitionsitic logic → picture logic: a brief reminder. We started with the usual fuzzy logic, in which, for each statement A with degree of confidence a_+, the degree of confidence in its negation $\neg A$ is equal to $a_- = 1 - a_+$, i.e., for which $a_+ + a_- = 1$.

We then took into account that we may be uncertainty about both A and $\neg A$, i.e., we may have $a_+ + a_- \le 1$. This is the case of intuitionistic fuzzy logic. In intuitionistic fuzzy logic, we can consider the remaining degree

$$a_0 \stackrel{\text{def}}{=} 1 - (a_+ + a_-)$$

that describes our uncertainty. In this case, we have $a_+ + a_0 + a_- = 1$.

The next natural step was to take into account that not all the difference $1 - (a_+ + a_-)$ may be due to uncertainty, and that we may have situations in which, for the corresponding uncertainty degree a_0, we have $a_0 < 1-(a_+ + a_-)$. In this case, in general, we have $a_0 \le 1 - (a_+ + a_-)$. This is the case of picture fuzzy logic. In picture fuzzy logic, we can consider the remaining degree $a_r \stackrel{\text{def}}{=} (1 - (a_+ + a_-)) - a_0$. In this case, we have $a_+ + a_0 + a_- + a_r = 1$.

Natural next steps. It seems reasonable to apply the same idea again, and consider the cases in which $a_+ + a_0 + a_- + a_r \le 1$. For this new extension of fuzzy logic, we can use the same idea as above to extend operations on a_+, a_-, and a_0 components to the operations of the remainders a_r. For example, for "and"-operations, we require that every time we have $a_+ + a_0 + a_- + a_r \le 1$ and $b_+ + b_0 + b_- + b_r \le 1$, we should have

$$f_\vee(a_+, b_+) + f_0(a_0, b_0) + f_\&(a_-, b_-) + f_r(a_r, b_r) \le 1,$$

and for each pair (a_r, b_r), we should have at least one case when the above inequality becomes equality.

From this requirement, we can also extract an explicit formula for $f_r(a, b)$: for example, for $f_\&(a, b) = \min(a, b)$, $f_\vee(a, b) = \max(a, b)$, and $f_0(a, b) = \min(a, b)$, we get $f_r(a, b) = \min(a, b)$.

For such extended logic, we can define a new remainder

$$a_n \stackrel{\text{def}}{=} 1 - (a_+ + a_0 + a_- + a_r)$$

for which $a_+ + a_0 + a_- + a_r = 1$. We can now again apply the same idea and consider cases for which $a_+ + a_0 + a_- + a_r \le 1$, etc. Our approach enables us to define "and"- and "or"-operations for all such extensions.

5 Proofs

Proof of Proposition 2.1. The first condition implies that

$$f(a_-, b_-) \le 1 - f_\&(a_+, b_+)$$

for all a_+ and b_+, and b_- for which $a_+ \le 1 - a_-$ and $b_+ \le 1 - b_-$. This means that $f(a_-, b_-)$ should be smaller than or equal to the smallest of the values $1 - f_\&(a_+, b_+)$ for such a_+ and b_+, i.e., equivalently, that is smaller than or equal to 1 minus the largest possible value of $f_\&(a_+, b_+)$.

Since "and"-operation is non-strictly increasing in both variables, its largest value is attained when both a_+ and b_+ attain their largest values $1 - a_-$ and $1 - b_-$. Thus, the first condition is equivalent to

$$f(a_-, b_-) \le 1 - f_\&(1 - a_-, 1 - b_-).$$

The second part of the definition implies that we have equality:

$$f(a_-, b_-) = 1 - \max_{a_+ \le 1 - a_-, b_+ \le 1 - b_-} f_\&(a_+, b_+).$$

The "and"-operation is monotonic in both variables, so, for the given values of a_- and b_-, the largest value of $f_\&(a_-, b_-)$ is attained when a_- and b_- attains the largest possible values $a_- = 1 - a_0 - a_+$ and $b_- = 1 - b_0 - b_+$. Thus, we get the desired formula. The proposition is proven.

Proof of Proposition 2.2 is similar.

Proof of Proposition 3.1. The first condition implies that

$$f_0(a_0, b_0) \le 1 - (f_\&(a_+, b_+) + f_\vee(a_-, b_-))$$

for all a_+, a_-, b_+, and b_- for which $a_+ + a_0 + a_- \le 1$ and $b_+ + b_0 + b_- \le 1$. Thus, we can conclude that

$$f_0(a_0, b_0) \le 1 - \max_{a_+ + a_0 + a_- \le 1, b_+ + b_0 + b_- \le 1} (f_\&(a_+, b_+) + f_\vee(a_-, b_-)).$$

The second part of the definition implies that we have equality:

$$f_0(a_0, b_0) = 1 - \max_{a_+ + a_0 + a_- \le 1, b_+ + b_0 + b_- \le 1} (f_\&(a_+, b_+) + f_\vee(a_-, b_-)).$$

The "or"-operation is monotonic in both variables, so, for the given values of a_+ and b_+, the largest value of $f_\vee(a_-, b_-)$ (and thus, of the sum $f_\&(a_+, b_+) + f_\vee(a_-, b_-)$) is attained when a_- and b_- attains the largest possible values $a_- = 1 - a_0 - a_+$ and $b_- = 1 - b_0 - b_+$. Thus, we get the desired formula. The proposition is proven.

Proof of Proposition 3.2. For $a_+ b_+ = 0$, we get

$$f_\&(a_+, b_+) + f_\vee(1 - a - a_+, 1 - b - b_-) = f_\&(0, 0) + f_\vee(1 - a, 1 - b) =$$

$$\min(0, 0) + \max(1 - a, 1 - b).$$

For all other values $a_+ \le 1 - a$ and $b_+ \le 1 - b$, we have

$$v \stackrel{\text{def}}{=} f_\&(a_+, b_+) + f(1 - a - a_+, 1 - b - b_+) =$$

$$\min(a_+, b_+) + \max(1 - a - a_+, 1 - b - b_+).$$

If we add the same constant c to two numbers p and q, the same number will remain larger, so we get $\max(p + c, q + c) = \max(p, q)$. In particular, we get

$$v = \max(1 - a - a_+ + \min(a_+, b_+), 1 - b - b_+ + \min(a_+, b_+)).$$

In the first of the two minimized terms, we can use the fact that $\min(a_+, b_+) \le a_+$, thus

$$1 - a - a_+ + \min(a_+, b_+) \le 1 - a - a_+ + a_+ = 1 - a.$$

In the second term, we can similarly use the fact that $\min(a_+, b_+) \le b_+$, thus

$$1 - b - b_+ + \min(a_+, b_+) \le 1 - b - b_+ + b_+ = 1 - b.$$

Hence, $v \le \max(1-a, 1-b)$. So, the maximum is indeed $\max(1-a, 1-b)$.

Thus, $f_0(a,b) = 1 - \max(1-a, 1-b)$. When the subtracted number is the largest, the difference is the smallest, so we get

$$f_0(a,b) = \min(1-(1-a), 1-(a-b)) = \min(a,b).$$

The proposition is proven.

Proof of Proposition 3.3. For $a_+ = b_+ = 0$, we get

$$f_\&(a_+, b_+) + f_\vee(1-a-a_+, 1-b-b_-) = f_\&(0,0) + f_\vee(1-a, 1-b) =$$

$$0 \cdot 0 + 1 - a + 1 - b - (1-a)\cdot(1-b) =$$

$$1 - a + 1 - b - 1 + a + b - a \cdot b = 1 - a \cdot b.$$

For all other values $a_+ \le 1-a$ and $b_+ \le 1-b$, we have

$$f_\&(a_+, b_+) + f(1-a-a_+, 1-b-b_+) =$$

$$a_+ \cdot b_+ + 1 - a - a_+ + 1 - b - b_+ - (1-a-a_+)\cdot(1-b-b_+) =$$

$$a_+ \cdot b_+ + 1 - a - a_+ + 1 - b - b_+ - 1 + a + a_+ + b + b_+ - a \cdot b - a \cdot b_+ - a_+ \cdot b - a_+ \cdot b_+ =$$

$$1 - a \cdot b - a \cdot b_+ - a_+ \cdot b \le 1 - a \cdot b.$$

Thus, the maximum is indeed $1 - a \cdot b$, and 1 minus this maximum is simply $a \cdot b$. The proposition is proven.

Proof of Proposition 3.4 is similar to the proof of Proposition 3.1.

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