Perfect Reproducibility Is Not Always Algorithmically Possible: A Pedagogical Observation

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Abstract

Users of computer-based based systems often want perfect reproducibility: when encountering the same situation twice, the system should exhibit the same behavior. For real-life systems that include sensors, this is not always possible: due to inevitable measurement uncertainty, for the same actual value of the corresponding quantity, we may get somewhat different measurement results, and thus, show somewhat different behavior. In this paper, we show that the above-described ideal reproducibility is not possible even in the idealized situation, when we assume that a sensor can perform its measurement with any given accuracy.

1 Formulation of the Problem

Users often want perfect reproducibility. Users of software and, more generally, users of computer-based systems often want perfect *reproducibility*: that if we repeatedly place the system in the exact same situation, it should react the exact same way.

In practice, perfect reproducibility may not be possible due to measurement errors. Of course, if the corresponding real-life system includes sensors and measurements, we cannot have exact reproducibility. Due to inevitable measurement errors, the value measured by the sensor may be different from the actual value of the corresponding quantity. Moreover, if we measure the same value several times, we may get different results; see, e.g., [1]. As a result, e.g., when we have a computer-controlled thermoregulation system, then even for the exact same temperature, the sensors readings will be slightly different and thus, the system's reaction may be slightly different.

At least the users want perfect reproducibility in the ideal situation. The above measurement uncertainty is well known, so what the users want is that the system's behavior be perfectly reproducible in the idealized situation, when we can measure each quantity with any given accuracy.

What we show in this paper. In this paper, we provide simple arguments that even in this idealized case, perfect reproducibility is not always possible.

2 Case Study and the Main Result

Description of the simple case study. Let us consider a very simple control situation, when we want to keep some quantity q at a given level q_0. To perform this task, we measure the actual value of q. As we have mentioned earlier, we consider an idealized case when for every integer n, we can measure q with accuracy of n binary digits (i.e., with an accuracy 2^{-n}). In such a measurement, we get a measurement result q_n which is 2^{-n}-close to q: $|q - q_n| \leq 2^{-n}$.

Let us also consider a very simplified version of a controller, with only two options:

- we can switch on a device that increases q,
- or we can switch on a device that decreases q,
- or we can do nothing.

One possible example of such a setting is a simple system for regulating the room temperature:

- if the temperature is above a certain threshold, we switch on the air conditioner,
- if the temperature is below a certain threshold, we switch on the heater, and
- if the room temperature is comfortable, we do nothing.

Another possible example may be a system for keeping a satellite at a given height above earth:

- if the height decreases, we switch on an engine that pushes the orbit up;
- if the height increases, we switch on another engine that pushes the orbit down; and
- if the height is close to its desired one, we do not do anything.

What control strategy we can apply. We would like to design a computer-based control system for this setting. This system can start by measuring the value of the desired quantity q with some initial accuracy of n_0 binary digits. Based on the result q_{n_0} of this measurement, we can make four possible decisions:

- we can switch on the device that increases q; we will denote the corresponding decision by $+$;
- we can switch on the device that decreases q; we will denote the corresponding decision by $-$;
- we can decide to do nothing at this point; we will denote the corresponding decision by 0; or
- we can select to perform a more accurate measurement; in this case, the system will generate an integer $n > n_0$, perform the measurement with accuracy 2^{-n}, and then, based on the new measurement result q_n, again select one of these four options.

After one or several iterations of this process, the control strategy has to eventually produce one of the three possible decisions: a plus decision $+$ (increase q), a minus decision $-$ (decrease q), or a 0 decision (do nothing).

It is also reasonable to assume that:

- when the value q is sufficiently large, i.e., when $q \geq \overline{q}$ for some threshold $\overline{q} > q_0$, then we should always make a minus decision, i.e., decision to decrease q, and
- when the value q is sufficiently small, i.e., when $q < \underline{q}$ for some threshold $\underline{q} < q_0$, then we should always make a plus decision, i.e., a decision to increase q.

Comment. In real life, we often have the option of performing a more accurate measurement. For example, if a person has fallen down and hurt himself, and an X-ray picture is inconclusive, a doctor may order an MRI image to get a more accurate picture of the damage. The main difference between such real-life situations and our idealized situation is that:

- in real life, there is always a limit of how accurately we can measure, while
- in our idealized setting, we assume that we can perform the measurement with an arbitrary accuracy.

Can we achieve perfect reproducibility in such a situation? The question is whether it is possible, in such an idealized situation, to achieve perfect reproducibility. In other words, is it possible to design a control strategy in such a way that for the same actual value of the parameter q, the system would make the exact same decision when this value is encountered the next time?

Our main result: formulation. In this paper, we prove that such a perfectly reproducible control algorithm is impossible.

Proof. We will prove this impossibility by contradiction. Let us assume that such a perfectly reproducible control strategy *is* possible. Then, for each actual value q of the corresponding quantity, this control algorithm returns one of the three control values: $+$, $-$, or 0.

For values $q \geq \overline{q}$, all the decisions are minus decisions. Thus, the set S_- of all the values q for which the algorithm produces a minus recommendation is non-empty.

For $q \leq \underline{q}$, all the decisions are $+$ recommendations. Thus, for these values q, we never make a minus decision. So, the S_- only contains values which are larger than $\underline{q}$. Therefore, the set S_- is bounded from below.

Since the set S_- is non-empty and bounded from below. Thus, this set has the greatest lower bound (infimum) $s \stackrel{\text{def}}{=} \inf(S_-)$.

One can easily see that for each n, in the 2^{-n}-vicinity of the value s, there exist:

- a point s_n^- for which the algorithm does not produce minus, and
- a point s_n^+ for which the algorithm does produce minus.

Indeed:

- as s_n^-, we can simply take $s_n^- = s - 2^{-n}$; since $s_n^- < s$, and s is the infimum of all the values for which the system returns minus, it cannot return minus for the value s_n^-;
- the existence of the value $s_n^+ \in S_-$ for which $s_n^+ \leq s + 2^{-n}$ is also easy to show: if there was no such s_n^+, this would mean that all the values from S_- are larger than $s + 2^{-n}$ and therefore, $s + 2^{-n}$ would be a lower bound for all the points from the set S_-. However, we know that s is the greatest lower bound, so the value $s + 2^{-n}$ which is larger than s cannot be a lower bound.

Now, let us analyze what exactly our algorithm is supposed to return when the actual value of the quantity q is exactly s. No matter what control recommendation the algorithm returns, this recommendation is based on the latest result of measuring q, and this measurement result q_N is accurate only with accuracy 2^{-N} for some N, i.e., $|q_N - s| \leq 2^{-N}$. Perfect reproducibility means that for the given value s, no matter the measurement result q_N is, we should make the same recommendation. Thus, we should produce the same recommendation for all measurement results q_N for which $|q_N - s| \leq 2^{-N}$, i.e., for all measurement results q_N from the interval $[s - 2^{-N}, s + 2^{-N}]$.

In situations in which the actual value is $s_N^- = s - 2^{-N}$, one of the possible measurement results is this same value $q_N = s_N^-$. Since this measurement result q_N is in the above interval $[s - 2^{-N}, s + 2^{-N}]$, this means that, based on this measurement result, we should make the same recommendation as for the value s.

- We know that for the value s_N^-, the recommendation is *not* minus, it is ether $+$ or 0.

- Thus, for the value s, we should produce the same recommendation of $+$ or 0.

On the other hand, in situations in which the actual value is $s_N^+ \in [s, s+2^{-N}]$, also one of the possible measurement results is this same value $q_N = s_N^+$. Since this value q_N is in the above interval $[s-2^{-N}, s+2^{-N}]$, this means that, based on this measurement result, we should make the same recommendation as for the value s.

- We know that for the value s_N^+, the recommendation *is* minus.
- Thus, for the value s, we should produce the same recommendation minus.

So, we get a contradiction:

- on the other hand, for the value s, the system should issue the recommendation of minus or 0;
- on the other hand, for the same value s, the system should issue the recommendation $+$.

This contradiction shows that a perfectly reproducible control strategy is indeed not possible.

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