Why Some Non-Classical Logics Are More Studied?

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Abstract It is well known that the traditional 2-valued logic is only an approximation to how we actually reason. To provide a more adequate description of how we actually reason, researchers proposed and studied many generalizations and modifications of the traditional logic, generalizations and modifications in which some rules of the traditional logic are no longer valid. Interestingly, for some of such rules (e.g., for law of excluded middle), we have a century of research in logics that violate this rule, while for others (e.g., commutativity of "and"), practically no research has been done. In this paper, we show that fuzzy ideas can help explain why some non-classical logics are more studied and some less studied: namely, it turns out that most studied are the violations which can be implemented by the simplest expressions (specifically, by polynomials of the lowest order).

1 Formulation of the Problem

Commonsense reasoning and formal logic. We all use logic in real life; we use phrases containing "and", "or", and "not" in our reasoning.

Since ancient times, researchers have been trying to describe such reasoning in precise terms – i.e., trying to transform commonsense reasoning into formal logic.

Traditional logic: a brief reminder. The most widely used formalization of logic is the traditional 2-valued logic, which was formally described by Boole in the 19th century. Operations of this logic have many well-known properties.

For example, for the "and"-operation $a \& b$ is:

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- "false" (0) and any a is equivalent to "false": $0\,\&\,a \Leftrightarrow 0$;
- similarly, a and "false" is equivalent to "false": $a\,\&\,0 \Leftrightarrow 0$;
- "true" (1) and any a is equivalent to a, as well as a and "true":

$$1\,\&\,a \Leftrightarrow a\,\&\,1 \Leftrightarrow a;$$

- this operation is *idempotent*, i.e., $a\,\&\,a$ is equivalent to a: $a\,\&\,a \Leftrightarrow a$;
- this operation is *commutative*, i.e., $a\,\&\,b$ is equivalent to $b\,\&\,a$:

$$a\,\&\,b \Leftrightarrow b\,\&\,a;$$

- this operation is *associative*, i.e., $a\,\&\,(a\,\&\,c)$ is equivalent to $(a\,\&\,b)\,\&\,c$:

$$a\,\&\,(b\,\&\,c) \Leftrightarrow (a\,\&\,b)\,\&\,c.$$

Similarly, the "or"-operation $a \vee b$ also satisfies similar properties:

- $0 \vee a \Leftrightarrow a \vee 0 \Leftrightarrow a$;
- $1 \vee a \Leftrightarrow a \vee 1 \Leftrightarrow 1$;
- it is idempotent, i.e., $a \vee a$ is equivalent to a;
- it is commutative, i.e., $a \vee b$ is equivalent to $b \vee a$, and
- it is associative, i.e., $a \vee (b \vee c)$ is equivalent to $(a \vee b) \vee c)$.

These two operations are distributive with respect to each other:

$$a\,\&\,(b \vee c) \Leftrightarrow (a \vee b)\,\&\,(a \vee c); \;\; a \vee (a\,\&\,c) \Leftrightarrow (a \vee b)\,\&\,(a \vee c).$$

Many properties of the traditional logic involve negation:

- the rule that $\neg(1)$ is 0, and that $\neg(1)$ is 0;
- the law of excluded middle, according to which $a \vee \neg a$ is always true;
- the double negation rule, according to which $\neg\neg a \Leftrightarrow a$; and
- de Morgan laws:

$$\neg(a\,\&\,b) \Leftrightarrow \neg a \vee \neg b; \;\; \neg(a \vee b) \Leftrightarrow \neg a\,\&\,\neg b.$$

The traditional 2-valued logic is only an approximation to commonsense reasoning. It is well known that boolean logic is only an approximation to how we actually reason. Our actual use of "and" and "or" is more complex.

For example, while in the formal logic, "and" is commutative, the phrases "I studied and I took the test" and "I took the test and I studied" clearly mean two different things.

Because of this difference, researchers have been trying to come up with extensions and modification of boolean logic that would better capture commonsense reasoning. As a result, we have a plethora of different non-classical logics; see, e.g., [3, 11].

Non-classical logics: examples. Since the early 20th century, many extensions and generalizations of classical logic have appeared:

- in some of these logics, propositional operations "and", "or", and "not" do not satisfy all the usual properties;
- other logics introduce additional propositional operations;
- yet other logics do both.

One of the first examples of non-classical logics was *intuitionistic logic* – developed early in the 20th century. This logic rejects the law of excluded middle, the law of double negation, and de Morgan laws.

Another example of a non-classical logic – which was known since Aristotle but which was formalized only in the early 20th century – was *modal logic*, that introduces additional unary operations "necessary" and "possible".

Linear logic rejects the rule $a\,\&\,a \Leftrightarrow a$.

Some logics use two different negation operations – usual negation and strong negation, etc.

Challenge. But why some extensions were developed, and some were not? For example, as we have mentioned earlier, in commonsense reasoning, "and" is not always commutative, but no mainstream logics seriously considers non-commutative "and"-operations. So why there have been developed logics rejecting some laws of boolean logic and not others?

What we do in this paper. At first glance, it may seem that the above challenge has no good answer – it is like asking why Picasso moved to a blue period and not to some other period. However, surprisingly, we do present an answer.

To come up with such an answer, we take into account yet another non-classical logic that was developed specifically to capture important features of commonsense reasoning – namely, fuzzy logic (see, e.g., [1, 4, 7, 8, 9, 10, 14]) in which:

- in addition to the traditional two truth values "true" (1) and "false" (0),
- we also allow intermediate truth values which are represented by numbers from the interval $[0,1]$.

It turns out that fuzzy ideas can indeed explain why some extensions of boolean logic have indeed been well-studied and some have not yet been thoroughly explored.

2 Main Idea Behind Our Explanation

Background. When we extend propositional operations from the 2-valued set $\{0,1\}$ to the interval $[0,1]$, each operation becomes a continuous function of the corresponding real-valued variables.

In general, such functions can be expanded in Taylor series – and can be thus approximated by polynomials, which correspond to keeping first few terms in this expansion.

As we increase the order of the corresponding polynomials, we get more and more accurate representation of the corresponding functions – but at the same time this representation becomes more complex.

Our main idea. It is natural to expect that:

- violations of logical laws that can be attained by the simplest (smallest order) functions will be explored first, while
- violations that require much higher (and thus, more complex) polynomials will be studied much later.

This idea helps explain why some non-classical logics are more studied. We will show that this natural idea helps explain why:

- some non-classical logics have been thoroughly studied, while
- others remain largely a not-well-studied idea.

3 Let Us Start Our Analysis: Simplest Polynomial Representations of "and", "or", and "not"

Let us start with "and". Let us consider the very first properties of an "and"-operation. Based on these properties, we have the following simple (and known) result.

Proposition 1.

- *There is no linear function $f(a,b)$ for which $f(0,a) = f(a,0) = 0$ and*
$$f(1,a) = f(a,1) = a.$$
- *The only quadratic function with these properties is $f(a,b) = a \cdot b$.*

Proof. Let us first consider the case of linear functions. A general linear function of two variables a and b has the form

$$f(a,b) = c_0 + c_1 \cdot a + c_2 \cdot b. \tag{1}$$

From $f(0,a) = 0$ we conclude that $c_0 + c_2 \cdot a = 0$ for all a. Thus, $c_0 = 0$ and $c_2 = 0$. Similarly, from $f(a,0) = 0$, we conclude that $c_1 = 0$ and thus, that $f(a,b) = 0$ for all a and b – while we wanted to have, e.g., $f(1,1) = 1 \neq 0$.

Similarly, a general quadratic function of two variables has the form

$$f(a,b) = c_0 + c_1 \cdot a + c_2 \cdot b + c_{11} \cdot a^2 + c_{12} \cdot a \cdot b + c_{22} \cdot b^2. \tag{2}$$

The requirement $f(0,a) = 0$ implies that $c_0 + c_2 \cdot a + c_{22} \cdot a^2 = 0$ for all a, hence $c_0 = c_2 = c_{22} = 0$. Similarly, the requirement $f(a,0) = 0$ implies that $c_1 = c_{11} = 0$.

Thus, $f(a,b) = c_{12} \cdot a \cdot b$. The condition $f(1,a) = a$ now implies that $c_{12} = 1$ and hence, that $f(a,b) = a \cdot b$.

The proposition is proven.

Comment. A similar result holds for "or"-operations.

Proposition 2.

- *There is no linear function $f(a,b)$ for which $f(0,a) = f(a,0) = a$ and*

$$f(1,a) = f(a,1) = 1.$$

- *The only quadratic function with these properties is $f(a,b) = a + b - a \cdot b$.*

Proof. Let us first consider the case of linear functions (1). From $f(0,a) = a$ we conclude that $c_0 + c_2 \cdot a = a$ for all a. Thus, $c_0 = 0$ and $c_2 = 1$. Similarly, from $f(a,0) = a$, we conclude that $c_1 = 1$ and, thus, that $f(a,b) = a+b$ for all a and b. So, for $a = 1$, we will get $f(1,1) = 1+1 = 2$, while we wanted to have $f(1,1) = 1 \neq 2$.

Similarly, for a quadratic function (2), the requirement $f(0,a) = a$ implies that $c_0 + c_2 \cdot a + c_{22} \cdot a^2 = a$ for all a, hence $c_0 = c_{22} = 0$ and $c_2 = 1$. Similarly, the requirement $f(a,0) = a$ implies that $c_{11} = 0$ and $c_1 = 1$. Thus,

$$f(a,b) = a + b + c_{12} \cdot a \cdot b.$$

The condition $f(1,a) = 1$ now implies that $1 + a + c_{12} \cdot a = 1$ and thus, that $c_{12} = -1$. So, $f(a,b) = a + b - a \cdot b$.

The proposition is proven.

Comment. For negation, a linear operation is possible.

Proposition 3. *The only linear function $f(a)$ for which $f(0) = 1$ and $f(1) = 0$ is the function $f(a) = 1 - a$.*

Proof is straightforward: it is based on the general form of a linear function of one variable:

$$f(a) = c_0 + c_1 \cdot a. \tag{3}$$

Discussion. What properties are satisfied for these simple operations: $a \cdot b$ for "and", $a + b - a \cdot b$ for "or", and $1 - a$ for "not"?

- The "and" and "or"-operations are commutative and associative.
- The law of double negation is satisfied.
- de Morgan laws are satisfied.

However, already for this simple example, we can see that two major laws are not satisfied:

- we do not have the excluded middle:

$$a \vee \neg a = 1 + (1 - a) - a \cdot (1 - a) = 1 - a \cdot (1 - a),$$

so this law is not always true; and
- in general, we have $a \& a = a^2 \neq a$, so $a \& a$ is not always equivalent to a.

Not surprisingly, the logics based on these violations – intuitionistic logic and linear logic – have been actively studied.

4 General Case of Quadratic Functions

Possible quadratic negation operations. In the previous section, we considered linear negation operations. However, since the only "and"- and "or"-operations are quadratic anyway, why not consider quadratic negation operations as well?

Proposition 3. *For quadratic functions $f(a)$, the following two conditions are equivalent to each other:*

- $f(0) = 1$ *and* $f(1) = 0$, *and*
- $f(a) = 1 - a - c_{11} \cdot a \cdot (1 - a)$.

Proof. For a general quadratic function of one variable

$$f(a) = c_0 + c_1 \cdot a + c_{11} \cdot a^2, \tag{4}$$

the condition $f(0) = 1$ implies that $c_0 = 1$, and the condition $f(1) = 0$ implies that $1 + c_1 + c_{11} = 0$, i.e., that $c_1 = -1 - c_{11}$. Thus, the expression (4) takes the form

$$f(a) = 1 - a - c_{11} \cdot a + c_{11} \cdot a^2 = 1 - a - c_{11} \cdot a \cdot (1 - a).$$

The proposition is proven.

Discussion.

- When $c_{11} \neq 0$, we have $f(f(a)) \neq a$. This explains why logics with no double negation property have been actively studied.
- We can have several different such operations, corresponding to different values c_{11}. This explains why logics with several different negation operations have been considered.
- The resulting function is not necessarily monotonic – this explains why non-monotonic logics have also been actively studied.

Other quadratic operations. What are the general extension of the identity function, i.e., a function for which $f(0) = 0$ and $f(1) = 1$?

Proposition 4.

- *The only linear function $f(a)$ for which $f(0) = 0$ and $f(1) = 1$ is the trivial function*

$$f(a) = a.$$

- *There exist non-trivial quadratic functions with the above properties; they all have the form* $f(a) = a - c_{11} \cdot a \cdot (1 - a)$.

Proof. The case of the linear function (3) is straightforward. Let us now consider the general case of a quadratic function (4). In this case, the requirement that $f(0) = 0$ implies that $c_0 = 0$, and the requirement that $f(1) = 1$ implies that $c_1 + c_{11} = 1$, thus $c_1 = 1 - c_{11}$, and the expression (4) takes the form

$$f(a) = a - c_{11} \cdot a + c_1 \cdot a^2 = a - c_{11} \cdot a \cdot (1 - a).$$

The proposition is proven.

Discussion. When $c_{11} > 0$, we have $f(a) \leq a$. This can be identified with the unary operation "necessary", for which the usual intuition is that:

- if something is absolutely true, it is also absolutely necessarily true: $f(1) = 1$;
- if something is absolutely false, it is also absolutely necessarily false: $f(0) = 0$;
- and in general, if something is necessarily true, then it is true – but not vice versa, so our degree of confidence that a statement is necessarily true can be smaller than our degree of confidence that it is true – it could be true accidentally.

When $c_{11} < 0$, we get always $x \leq f(x)$. This can be identified with unary operation "possible" in modal logic.

So it is not surprising that modal logic have been actively developed. It should also be noticed that, in general, for $c_{11} \neq 0$, we have $f(f(a)) \neq a$. This explains why modal logic actively studies logics in which an iteration of necessity is not equivalent to a single necessity operation.

5 What About Cubic Operations?

To explore the possibility of violating other rules, let us consider cubic "and"- and "or"-operations, i.e., functions of the type

$$f(a,b) = c_0 + c_1 \cdot a + c_2 \cdot b + c_{11} \cdot a^2 + c_{12} \cdot a \cdot b + c_{22} \cdot b^2 + \\ c_{111} \cdot a^3 + c_{112} \cdot a^2 \cdot b + c_{122} \cdot a \cdot b^2 + c_{222} \cdot b^3. \quad (5)$$

Proposition 5. *The only cubic function for which* $f(0,a) = f(a,0) = 0$ *and* $f(1,a) = f(a,1) = a$ *is* $f(a,b) = a \cdot b$.

Proof. For the expression (5), the condition $f(0,b) = 0$ implies that

$$c_0 + c_2 \cdot b + c_{22} \cdot b^2 + c_{222} \cdot b^3 = 0$$

for all b. Thus, $c_0 = c_2 = c_{22} = c_{222} = 0$.

Similarly, the condition $f(a,0) = 0$ implies that $c_1 = c_{11} = c_{111} = 0$. Thus, the expression (5) takes the form

$$f(a,b) = c_{12} \cdot a \cdot b + c_{112} \cdot a^2 \cdot b + c_{122} \cdot a \cdot b^2.$$

For this formula, the requirement that $f(1,b) = b$ implies that

$$c_{12} \cdot b + c_{112} \cdot b + c_{122} \cdot b^2 = 0$$

for all b, hence $c_{122} = 0$. Similarly, the requirement that $f(a,1) = a$ implies that

$$c_{112} = 0.$$

Thus, all cubic terms are 0, so $f(a,b)$ is actually a quadratic function, and for quadratic functions, we already know that the only operation with the desired properties is $f(a,b) = a \cdot b$.

The proposition is proven.

Proposition 6. *The only cubic function for which $f(0,a) = f(a,0) = a$ and $f(1,a) = f(a,1) = 1$ is $f(a,b) = a + b - a \cdot b$.*

Proof. We can prove this result by taking into account that, as one can easily prove, a function $f(a,b)$ satisfies the corresponding conditions if and only if its "dual"

$$g(a,b) = 1 - f(1-a, 1-b)$$

satisfies the conditions of Proposition 5. Since we already know, from Proposition 5, that $g(a,b) = a \cdot b$, we can thus conclude that

$$f(a,b) = 1 - g(1-a, 1-b) = 1 - (1-a) \cdot (1-b) = a + b - a \cdot b.$$

The proposition is proven.

Discussion. So, even if we consider cubic terms, we will still get only commutative and associative "and"- and "or"-operations. In view of our general approach, this explains why in the vast majority of logics studied so far, these operations are indeed commutative and associative.

To find example of non-commutative and/or non-associative logics, we therefore need to go to polynomials of even higher orders.

6 Case of 4th Order Operations

Let us consider general 4th order functions

$$f(a,b) = c_0 + c_1 \cdot a + c_2 \cdot b + c_{11} \cdot a^2 + c_{12} \cdot a \cdot b + c_{22} \cdot b^2 +$$

$$c_{111} \cdot a^3 + c_{112} \cdot a^2 \cdot b + c_{122} \cdot a \cdot b^2 + c_{222} \cdot b^3 +$$

$$c_{1111} \cdot a^4 + c_{1112} \cdot a^3 \cdot b + c_{1122} \cdot a^2 \cdot b^2 + c_{1222} \cdot a \cdot b^3 + c_{2222} \cdot b^4. \quad (6)$$

Proposition 7. *For 4th order functions, the following two conditions are equivalent to each other:*

- *for all a, we have $f(0,a) = f(a,0) = 0$ and $f(1,a) = f(a,1) = a$, and*
- *the function $f(a,b)$ has the form*

$$f(a,b) = a \cdot b - c \cdot a \cdot (1-a) \cdot b \cdot (1-b)$$

for some c.

Proof. The condition $f(0,b) = 0$ implies that for all b, we have

$$c_0 + c_2 \cdot b + c_{22} \cdot b^2 + c_{222} \cdot b^3 + c_{2222} \cdot b^4 = 0.$$

Thus, we have

$$c_0 = c_2 = c_{22} = c_{222} = c_{2222} = 0.$$

Similarly, the condition that $f(a,0) = 0$ for all a implies that

$$c_1 = c_{11} = c_{111} = c_{1111} = 0.$$

Thus, the general formula (6) gets the following simplified form:

$$f(a,b) =$$

$$c_{12} \cdot a \cdot b + c_{112} \cdot a^2 \cdot b + c_{122} \cdot a \cdot b^2 + c_{1112} \cdot a^3 \cdot b + c_{1122} \cdot a^2 \cdot b^2 + c_{1222} \cdot a \cdot b^3.$$

For this function, the requirement that $f(1,b) = b$ for all b implies that

$$c_{12} \cdot b + c_{112} \cdot b + c_{122} \cdot b^2 + c_{1112} \cdot b + c_{1122} \cdot b^2 + c_{1222} \cdot b^3 = b.$$

Thus, $c_{1222} = 0$. Similarly, the condition $f(a,1) = a$ implies that $c_{1112} = 0$. Thus, the above equality takes the following simplified form:

$$(c_{12} + c_{112}) \cdot b + (c_{122} + c_{1122}) \cdot b^2 = b.$$

So, $c_{12} + c_{112} = 1$, hence $c_{112} = 1 - c_{12}$, and $c_{1112} = -c_{112}$. So, if we denote $c \stackrel{\text{def}}{=} 1 - c_{12}$, we get $c_{112} = c$, $c_{12} = 1 - c$, and $c_{1122} = -c$. Similarly, from the condition that $f(a,1) = a$, we conclude that $c_{122} = c$. Thus, we get the desired expression for the function:

$$f(a,b) = a \cdot b - c \cdot (a \cdot b - a^2 \cdot b - a \cdot b^2 + a^2 \cdot b^2) = a \cdot b \cdot (1 - c \cdot (1 - a - b + a \cdot b)) =$$

$$a \cdot b(1 - c \cdot (1-a) \cdot (1-b)) = a \cdot b - c \cdot a \cdot (1-a) \cdot b \cdot (1-b).$$

The proposition is proven.

Comment. For "or"-operations we have a similar result.

Proposition 8. *For 4th order functions, the following two conditions are equivalent to each other:*

- *for all a, we have* $f(0,a) = f(a,0) = a$ *and* $f(1,a) = f(a,1) = 1$*, and*
- *the function* $f(a,b)$ *has the form*

$$f(a,b) = a + b - a \cdot b - c \cdot a \cdot (1-a) \cdot b \cdot (1-b)$$

for some c.

Proof. Similar to the proof of Proposition 6, we can use the fact that the dual function $g(a,b) = 1 = f(1-a, 1-b)$ satisfies the conditions of Proposition 7 and thus, has the form

$$f(a,b) = a \cdot b - c \cdot a \cdot (1-a) \cdot b \cdot (1-b).$$

Thus,

$$g(a,b) = 1 - f(1-a, 1-b) = 1 - (1-a) \cdot (1-b) + c \cdot a \cdot (1-a) \cdot b \cdot (1-b).$$

By taking into account that

$$1 - (1-a) \cdot (1-b) = a + b - a \cdot b,$$

we get the desired expression. The proposition is proven.

Discussion. We did not assume neither commutativity nor associativity. Interestingly, we got operations which are commutative but *not* associative. This explains why at least some research has been done for non-associative logics (see, e.g., [2, 5, 6, 12, 13] and references therein), while much fewer results are known for non-commutative ones.

To get non-commutative operations, we need to consider at least 5th order polynomials. For 5th order polynomials, it is already possible to have a non-commutative operation: for example, we can take

$$f(a,b) = a \cdot b - (c_a \cdot a + c_b \cdot b) \cdot a \cdot (1-a) \cdot b \cdot (1-b),$$

for any $c_a \neq c_b$.

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References

1. R. Belohlavek, J. W. Dauben, and G. J. Klir, *Fuzzy Logic and Mathematics: A Historical Perspective*, Oxford University Press, New York, 2017.
2. B. Bouchon-Meunier, V. Kreinovich, and H. T. Nguyen, "Non-associative operations", *Proceedings of the Second International Conference on Intelligent Technologies InTech'2001*, Bangkok, Thailand, November 27–29, 2001, pp. 39–46.
3. D. M. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic*, Springer, Cham, Switzerland, 2018.
4. G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall, Upper Saddle River, New Jersey, 1995.
5. V. Kreinovich, "Towards more realistic (e.g., non-associative) 'and'- and 'or'-operations in fuzzy logic", *Soft Computing*, 2004, Vol. 8, No. 4, pp. 274–280.
6. J. Martinez, L. Macias, A. Esper, J. Chaparro, V. Alvarado, S. A. Starks, and V. Kreinovich, "Towards more realistic (e.g., non-associative) and- and or-operations in fuzzy logic", *Proceedings of the 2001 IEEE Systems, Man, and Cybernetics Conference*, Tucson, Arizona, October 7–10, 2001, pp. 2187–2192.
7. J. M. Mendel, *Uncertain Rule-Based Fuzzy Systems: Introduction and New Directions*, Springer, Cham, Switzerland, 2017.
8. H. T. Nguyen and V. Kreinovich, "Nested intervals and sets: concepts, relations to fuzzy sets, and applications", In: R. B. Kearfott and V. Kreinovich (eds.), *Applications of Interval Computations*, Kluwer, Dordrecht, 1996, pp. 245–290.
9. H. T. Nguyen, C. Walker, and E. A. Walker, *A First Course in Fuzzy Logic*, Chapman and Hall/CRC, Boca Raton, Florida, 2019.
10. V. Novák, I. Perfilieva, and J. Močkoř, *Mathematical Principles of Fuzzy Logic*, Kluwer, Boston, Dordrecht, 1999.
11. G. Priest, *An Introduction to Non-Classical Logic: From If to Is*, Cambridge University Press, Cambridge, UK, 2008.
12. R. Trejo, V. Kreinovich, I. R. Goodman, J. Martinez, and R. Gonzalez, "A realistic (non-associative) logic and a possible explanations of 7 ± 2 law", *International Journal of Approximate Reasoning*, 2002, Vol. 29, pp. 235–266.
13. G. Xiang and V. Kreinovich, "Towards improved trapezoidal approximation to intersection (fusion) of trapezoidal fuzzy numbers: specific procedure and general non-associativity theorem", *Proceedings of the IEEE World Congress on Computational Intelligence WCCI'2010*, Barcelona, Spain, July 18–23, 2010, pp. 3120–3125.
14. L. A. Zadeh, "Fuzzy sets", *Information and Control*, 1965, Vol. 8, pp. 338–353.