

How to Use Quantum Computing to Check Which Inputs Are Relevant: A Proof That Deutsch-Jozsa Algorithm Is, In Effect, the Only Possibility

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Abstract—One of the main reasons why computations – in particular, engineering computations – take long is that, to be on the safe side, models take into account all possible affecting features, most of which turn out to be not really relevant for the corresponding physical problem. From this viewpoint, it is desirable to find out which inputs are relevant. In general, the problem of checking the input’s relevancy is itself NP-hard, which means, crudely speaking, that no feasible algorithm can always solve it. Thus, it is desirable to speed up this checking as much as possible. One possible way to speed up such a checking is to use quantum computing, namely, to use the Deutsch-Jozsa algorithm. However, it is not clear whether a more efficient quantum algorithm is possible for solving this problem. In this paper, we show that the Deutsch-Jozsa algorithm is, in effect, the only possible way to use quantum computing for checking which inputs are relevant – so no alternative quantum algorithms are possible.

Keywords—Detecting relevant inputs; NP-hardness; quantum computing; Deutsch-Jozsa algorithm

I. FORMULATION OF THE GENERAL PROBLEM

Need to speed up data processing. In spite of the fantastic advances in computer technologies, in spite of the fact that computer processing now is several orders of magnitude faster than it was in the past, there are still many computational problems – in engineering and in other applications – whose solution takes too long a time, even on the fastest high performance computers.

What can we do about it – other than designing faster computers? The need to speed up data processing is a well-recognized need that motivates many efforts to design even faster computers.

But even for the existing computers, there is a usually a way to speed them up. This possibility comes from the fact that when we set up time-consuming simulations of real-life processes – be it simulations of atmospheric processes in meteorology or simulations of molecular interactions in biomedical applications – we do not a priori know which inputs are relevant and which are not. As a result, in our simulations, we use all the inputs that *may* be relevant. Because of this, a large number of these inputs are actually *not* relevant.

If we could figure out which inputs are relevant and which are not relevant, and limit ourselves only to relevant inputs, we would be able to save a lot of computation time which is now wasted on processing irrelevant inputs and thus, speed up the corresponding computations.

Need to check which inputs are relevant is well understood.

This need to separate relevant from irrelevant inputs is well-recognized in physics (see, e.g., [2], [9]): it is an important skill of a physicist, enabling physicists to find solutions to complex physics-related equations much faster and much easier than mathematicians who do not have this skill. A classical example comes from General Relativity:

- the famous mathematician David Hilbert came with these equations practically at the same time as Einstein (his paper describing these equations was submitted only two weeks later than Einstein’s);
- however, equations is all Hilbert did, while
- Einstein, undeterred by the complexity of these nonlinear partial differential equations, also provided observable consequences – based on his ability to ignore what could be ignored in the corresponding physical situation and to consider only the most relevant inputs.

Checking which inputs are relevant is, in general, a difficult computational problem.

In general, the problem of checking which inputs are relevant and which are not is a probably difficult computational problem. Even for the simple case when the inputs are 1-bit (boolean) variables x_1, \dots, x_n , and the data processing algorithm consists of applying some propositional (boolean) operations (“and”, “or”, and “not”) to these variables, the problem of checking whether these inputs are needed at all or the result is always false (0) is a known NP-hard problem (see, e.g., [7]): it is, in effect, the same problem as the known NP-hard problem of checking whether a given propositional formula $f(x_1, \dots, x_n)$ can be satisfied, i.e., whether there exist values x_1, \dots, x_n that make it true.

Not only this problem of checking which inputs are relevant is difficult, it is the most difficult of all the problems: NP-hardness means that if we can solve this problem is feasible

time, then we can solve *any* problem with easy-to-check solution in feasible time.

In practice, this means that algorithms for checking which inputs are relevant themselves require a lot of computation time. Thus, we need to speed up these algorithms as well.

How can we speed up checking of which inputs are relevant. In general, if the *existing* technology does not enable us to compute something sufficiently fast, a natural way to speed up computations is to use *new* technology, a technology that utilizes additional physical processes. Computers consist of many very small parts, and for such very small objects, many physical processes involve the use of quantum physics – the physics that describes micro-objects like molecules, atoms, elementary particles, etc. Indeed, the idea of using quantum effects in computing turned out to be very successful; see, e.g., [3], [4], [6], [8].

In particular, there exists an efficient quantum algorithm – Deutsch-Josza algorithm – that checks whether a given bit is relevant for computations; see, e.g., [1], [5], [6].

Formulation of the problem. The challenge is that while the Deutsch-Josza algorithm is efficient, it is not clear whether this is the only possible quantum algorithm for solving this problem – maybe a better quantum algorithm is possible?

What we do in this paper. In this paper, for the simplest case of a 1-bit input, we show that the Deutsch-Josza algorithm is, in effect, the only possible quantum algorithm for solving this problem.

II. HOW TO CHECK WHETHER AN INPUT IS RELEVANT: 1-D CASE OF THE GENERAL PROBLEM

Formulation of the problem. In the 1-bit case, we are given a function $f(x)$ of one bit. In the non-quantum case, this means that we are given a black box that transforms a 1-bit state x into a new 1-bit state $f(x)$. We want to check whether the input x is relevant.

If $f(0) = f(1)$, then the result of applying the function $f(x)$ does not depend on the input, and the input is thus irrelevant. On the other hand, if $f(0) \neq f(1)$, this means that the value $f(x)$ depends on the input, so the input is relevant. So, checking whether the input is relevant means checking whether $f(0) \neq f(1)$.

How this problem can be solved in the case of non-quantum computing. In non-quantum computing, the only way to use the black box for computing f is to apply this box either to 0 or to 1. Thus, to check whether $f(0) = f(1)$, we need to call the algorithm f *twice*:

- once for $x = 0$ and
- the second time for $x = 1$.

It is desirable to speed up computation. Since, as we have mentioned earlier, $f(x)$ may be a very complex time-consuming algorithm, the need to run it twice requires too much time.

The desired speed-up can be achieved in quantum computing. It turns out that in quantum computing, we can check the equality $f(0) = f(1)$ by calling the function f only once.

To describe how this can be done, let us briefly recall what is quantum computing in the first place.

III. QUANTUM COMPUTING: A BRIEF REMINDER

States in quantum mechanics. Let us first recall the main ideas behind quantum computing. In quantum physics, in addition to the usual states s_1, \dots, s_n , it is also possible to form a *superposition*, i.e., a new state

$$s = a_1 s_1 + \dots + a_n s_n,$$

where a_i are complex numbers for which

$$|a_1|^2 + \dots + |a_n|^2 = 1.$$

Quantum measurements. If we apply, to the above superposition state, the usual measurement procedure that checks which of the states s_i we are in, we will get the state s_i with probability $|a_i|^2$.

These probabilities should add up to one – which explains the above restriction on the coefficients a_i .

The notion of a qubit. In particular, for a usual 1-bit state (0 or 1), in addition to the traditional states – which in quantum physics are denoted by $|0\rangle$ and $|1\rangle$ – we can also have superposition states $a_0|0\rangle + a_1|1\rangle$. The corresponding quantum analogue of a bit is known as a *quantum bit*, or *qubit*, for short.

States of multi-particle/multi-bit systems. When the object of study consists of two independent parts, then in classical physics, the state of the object can be described by describing the states s_i and s'_j of each part. Similarly, in quantum physics, if we have two independent parts, one in a state

$$s = a_1 s_1 + \dots + a_n s_n$$

and another in a state

$$s' = a'_1 s'_1 + \dots + a'_m s'_m,$$

then the whole object is in the state

$$(a_1 \cdot a'_1) |s_1 s'_1\rangle + (a_1 \cdot a'_2) |s_1 s'_2\rangle + \dots + (a_n \cdot a'_m) |s_n s'_m\rangle.$$

This state is called a *tensor product* of the states s and s' and is usually denoted by $s \otimes s'$. In this paper, we will use the tensor product of two qubits.

Measuring 2-qubit states. For a 2-qubit state, if we measure the state of one of the qubits – e.g., the first one – then the state of the first bit changes to 0 or 1, and the resulting state of the 2-bit system is *normalized*: the corresponding coefficients are divided by a constant so that the sum of the squares of absolute values remain to be 1. For example, if we measure the value of the first bit in the state

$$\frac{1}{2} |00\rangle - \frac{1}{2} |01\rangle + \frac{1}{2} |10\rangle - \frac{1}{2} |11\rangle,$$

and the result of this measurement is 0, then we take the remaining state

$$\frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle$$

and normalize it by multiplying it by $\sqrt{2}$; the resulting state is

$$\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|01\rangle.$$

Transformations of quantum states: general idea. We can also perform some linear transformations on the set of all possible quantum states – provided that these transformation preserve the equality $\sum_{i=1}^n |a_i|^2 = 1$. This is equivalent to requiring that the states corresponding to orthogonal vectors $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ (i.e., vectors for which the dot product $a \cdot b = \sum_{i=1}^n a_i \cdot b_i^* = 0$, where b_i^* means complex conjugate) get transformed into orthogonal ones.

Hadamard transformation. A typical example of such a transformation is a so-called *Hadamard transformation* H that transforms $|0\rangle$ into

$$H|0\rangle = \frac{1}{\sqrt{2}} \cdot |0\rangle + \frac{1}{\sqrt{2}} \cdot |1\rangle$$

and $|1\rangle$ into

$$H|1\rangle = \frac{1}{\sqrt{2}} \cdot |0\rangle - \frac{1}{\sqrt{2}} \cdot |1\rangle.$$

One can easily check that if we apply H twice, we get back the same state: $H(H|0\rangle) = |0\rangle$ and $H(H|1\rangle) = |1\rangle$.

How functions are represented in quantum computing. The last thing we need to describe quantum computing is how functions are represented.

The problem is that many useful functions are not reversible: e.g., if we know that $f(a, b) = a \& b$ is false, we cannot uniquely determine the values of a and b , they can be both false, or one of them can be false and another true. On the other hand, on the microlevel of quantum physics, all operations are reversible.

So, to represent a function $f(x_1, \dots, x_n)$ from bits to bits in quantum physics, we represent it as a reversible transformation

$$|x_1, \dots, x_n, y\rangle \rightarrow |x_1, \dots, x_n, y \oplus f(x_1, \dots, x_n)\rangle,$$

where $a \oplus b$ is exclusive or, which is the same as addition modulo 2.

Such transformations are reversible: indeed, if we apply the same transformation again, the first n bits do not change, while the last bit becomes

$$(y \oplus f(x_1, \dots, x_n)) \oplus f(x_1, \dots, x_n) =$$

$$y \oplus (f(x_1, \dots, x_n) \oplus f(x_1, \dots, x_n)) = y,$$

since for addition modulo 2, we always have $a \oplus a = 0$.

Now, we are ready to describe the Deutsch-Josza quantum algorithm for solving this problem.

IV. DEUTCH-JOSZA ALGORITHM: REMINDER.

What is the input to this algorithm. We are given a function $f(x)$ of one bit, i.e., a black box that transform a 2-bit state $|x, y\rangle$ into a new state $|x, y \oplus f(x)\rangle$.

What is the desired output. We want to check whether the input x is relevant, i.e., whether $f(0) \neq f(1)$.

How well this algorithm performs. While in non-quantum computing, we need at least two calls to f to check whether $f(0) = f(1)$, in quantum computing, the Deutsch-Josza quantum algorithm can check the equality $f(0) = f(1)$ by calling the function f only once.

Preliminary step. First, a preliminary step: we start with a state $|01\rangle = |0\rangle \otimes |1\rangle$ and apply the Hadamard transformation to both bits. As a result, we get the following state:

$$\begin{aligned} H(|0\rangle) \otimes H(|1\rangle) &= \\ \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \right) &= \\ \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle - \frac{1}{2}|11\rangle. \end{aligned}$$

The remaining step. After the preliminary step, we do the following:

- we apply f to the quantum state resulting from the preliminary step;
- then, we again apply the Hadamard transformation to both bits;
- after that, we measure the state of the first bit.

Based on the result of this measurement, we inform the user where the given function f is a constant, i.e., whether the input is relevant:

- If the resulting state of the first bit is 0, we conclude that the function $f(x)$ is constant, i.e., that the input is *not* relevant.
- If the resulting state of the first bit is 1, we conclude that the function $f(x)$ is not constant, i.e., that the input *is* relevant.

Proof of correctness: general idea. To prove the algorithm's correctness, let us consider all possible bit-to-bit functions $f(x)$.

We have two possible values for $f(0)$, for each of which we have two possible values of $f(1)$. Thus, overall, we have four possible cases:

- the case when $f(0) = 0$ and $f(1) = 0$;
- the case when $f(0) = 0$ and $f(1) = 1$, i.e., when $f(x) = x$ for all x ;
- the case when $f(0) = 1$ and $f(1) = 0$, i.e., when $f(x) = \neg x$ for all x ; and
- the case when $f(0) = 1$ and $f(1) = 1$.

Let us consider these four cases one by one.

Case when $f(0) = f(1) = 0$. In this case, after applying the function f , we get $y' = y \oplus f(x) = y$. So the state does not

change, and when we apply the Hadamard transform again, the state gets back to $|01\rangle$. So, the first bit is in 0 state.

Case when $f(0) = f(1) = 1$. In this case, we get $y' = y \oplus f(x) = y \oplus 1$. Here 0 is changed to 1 and 1 is changed to 0. As a result the state of the 2-bit system changes to

$$\frac{1}{2}|01\rangle - \frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle - \frac{1}{2}|10\rangle.$$

One can check that when we apply the Hadamard transform again, the state of the system changes to $-|01\rangle$. Here also, the first bit is in the 0 state.

Case when $f(x) = x$. In this case, the application of f leads to:

$$f(|00\rangle) = |00\rangle, f(|01\rangle) = |01\rangle, f(|10\rangle) = |11\rangle, \\ \text{and } f(|11\rangle) = |10\rangle.$$

Thus, the superposition state changes to

$$\frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|11\rangle - \frac{1}{2}|10\rangle.$$

When we apply the Hadamard transformation to this state, we get $|11\rangle$. Here, the first bit is in the 1 state.

Case when $f(x) = \neg x$. Finally, let us consider the case when $f(x) = \neg x$. In this case,

$$f(|00\rangle) = |01\rangle, f(|01\rangle) = |00\rangle, f(|10\rangle) = |10\rangle, \\ \text{and } f(|11\rangle) = |11\rangle,$$

so the superposition changes to the following state:

$$\frac{1}{2}|01\rangle - \frac{1}{2}|00\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle.$$

When we apply the Hadamard transformation to this state, we get $-|11\rangle$, so the first bit is 1.

Summarizing. In all four cases:

- when the function is constant, the algorithm returns 0, and
- when the function is not constant, the algorithm returns 1.

Thus, Deutsch-Josza algorithm indeed solves our original problem – and solves it in just one call to f instead of two.

Let us now prove that no other quantum algorithm can solve this problem.

V. PROOF THAT DEUTSCH-JOSZA ALGORITHM IS, IN EFFECT, THE ONLY POSSIBLE QUANTUM ALGORITHM FOR CHECKING WHETHER THE INPUT IS RELEVANT

General scheme. We want an algorithm that calls f only once. Thus, we first perform some transformations, resulting in some 2-qubit state

$$a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle.$$

To this state, we apply the function f , and then we again apply some transformations to each bit. After all this, we expect to get 0 if the input is irrelevant and 1 if it is relevant.

In the case of $f(x) = 0$, the application of f does not change the state, i.e., we get the same state

$$(a_{00}|0\rangle + a_{10}|1\rangle) \otimes |0\rangle + (a_{01}|0\rangle + a_{11}|1\rangle) \otimes |1\rangle.$$

No matter what we do with the second bit, the first bit needs to get into the 0 state, so we have

$$a_{00}|0\rangle + a_{10}|1\rangle \rightarrow |0\rangle \text{ and } a_{01}|0\rangle + a_{11}|1\rangle \rightarrow |0\rangle.$$

All quantum transformations are reversible. Thus, the fact that these two states of the first bit get transformed into the same state $|0\rangle$ means that these states are identical, i.e., that

$$a_{00}|0\rangle + a_{10}|1\rangle = C \cdot (a_{01}|0\rangle + a_{11}|1\rangle)$$

for some normalizing coefficient C . Thus, we conclude that

$$\frac{a_{01}}{a_{00}} = \frac{a_{11}}{a_{10}}. \quad (1)$$

For $f(x) = x$, applying the function f leads to the state

$$a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|11\rangle + a_{11}|10\rangle = \\ (a_{00}|0\rangle + a_{11}|1\rangle) \otimes |0\rangle + (a_{01}|0\rangle + a_{10}|1\rangle) \otimes |1\rangle.$$

Here, the first bit needs to go into the 1 state, i.e., we have

$$a_{00}|0\rangle + a_{11}|1\rangle \rightarrow |1\rangle \text{ and } a_{01}|0\rangle + a_{10}|1\rangle \rightarrow |1\rangle.$$

Here also, the fact that these two states of the first bit get transformed into the same state $|1\rangle$ means that these states are identical, i.e., that

$$a_{00}|0\rangle + a_{11}|1\rangle = C \cdot (a_{01}|0\rangle + a_{10}|1\rangle)$$

for some normalizing coefficient C . Thus, we conclude that

$$\frac{a_{01}}{a_{00}} = \frac{a_{10}}{a_{11}}. \quad (2)$$

Comparing equalities (1) and (2), we conclude that for $x \stackrel{\text{def}}{=} \frac{a_{11}}{a_{10}}$, we have $x = 1/x$, hence $x^2 = 1$ and so, $x = \pm 1$.

The ratio x cannot be equal to 1, since then we would have $a_{10} = a_{11}$, but we have

$$a_{00}|0\rangle + a_{10}|1\rangle \rightarrow |0\rangle \text{ and } a_{00}|0\rangle + a_{11}|1\rangle \rightarrow |1\rangle,$$

so $a_{10} \neq a_{11}$. Thus, we must have $x = -1$.

So, we have

$$a_{00}|0\rangle + a_{10}|1\rangle \rightarrow |0\rangle \text{ and } a_{00}|0\rangle - a_{10}|1\rangle \rightarrow |1\rangle.$$

The states $|0\rangle$ and $|1\rangle$ are orthogonal. So, by the properties of quantum transformations, the states $a_{00}|0\rangle + a_{10}|1\rangle$ and $a_{00}|0\rangle - a_{10}|1\rangle$ must also be orthogonal, i.e., we must have

$$a_{00} \cdot a_{00}^* - a_{10} \cdot a_{01}^* = |a_{00}|^2 - |a_{10}|^2 = 0.$$

Thus, $|a_{10}| = |a_{00}|$, and $a_{10} = \exp(i \cdot \varphi) \cdot a_{00}$ for some angle φ .

From the fact that the probabilities should add up to 1, we conclude that $4|a_{00}|^2 = 1$, hence $|a_{00}| = 1/2$. In quantum physics, states differing only by a complex factor whose absolute value is 1 are considered identical. Thus, we can safely assume that $a_{00} = 1/2$. Hence, $a_{01} = -a_{00} = -1/2$.

Then, $a_{10} = \exp(i \cdot \varphi) \cdot (1/2)$ and $a_{11} = -a_{10}$. So, for the original 2-bit state, we get the following expression:

$$\frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{\exp(i \cdot \varphi)}{\sqrt{2}}|10\rangle - \frac{\exp(i \cdot \varphi)}{2}|11\rangle.$$

This is almost the same as for the original Deutsch-Josza algorithm: the only minor difference is the factor $\exp(i \cdot \varphi)$ that does not affect any probabilities.

Modulo this minor difference, the Deutsch-Josza algorithm is indeed, the only possible algorithm for solving the problem – no alternative solution is possible. Our result has been proven.

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