Why The Obvious Necessary Condition is (Often) Also Sufficient (TONCAS): An Explanation of the Phenomenon

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Abstract

In many graph-related problems, an obvious necessary condition is often also sufficient. This phenomenon is so ubiquitous that it was even named TONCAS, after the first letters of the phrase describing this phenomenon. In this paper, we provide a possible explanation for this phenomenon.

1 Formulation of the Problem

TONCAS phenomenon: an example. When is a graph planar? In precise terms, when can a graph be embedded in a plane, i.e., represented by a graph in which edges intersect only at vertices? Clearly, if a graph contains a subgraph K_5 is which all five vertices are connected to each other, it cannot be embedded into a plane. Similarly, a planar graph cannot contain a subgraph $K_{3,3}$ that has two groups of 3 vertices, so that each of each vertex from the first group is connected with each vertex from the second group. Interesting, these two necessary conditions are sufficient: if a graph does not contain any subgraphs isomorphic to K_5 or to $K_{3,3}$, then this graph is planar; see, e.g., [7].

Another known example is checking whether a given lattice is distributive: this is equivalent to requiring that no sublattice is isomorphic to one of the prohibitive 5- and 6-vertices sublattices; see, e.g., [1].

TONCAS: general phenomenon. It turns out that this phenomenon is ubiquitous in graph theory and in related areas: in many such cases, the obvious necessary condition is also sufficient. This is known as the TONCAS phenomenon, after the first letters of the words that describe this phenomenon.

Why? A natural question is: why? How can we explain that for many natural properties, we have this phenomenon?

2 Analysis of the Problem

Let us formulate the TONCAS phenomenon in general terms. For simplicity, let us consider properties like embedding-related ones, in which, if the entire graph has this property, then all its subgraphs also satisfy the same property. In particular, if a graph satisfies the given property, then all its subgraphs with n or fewer vertices has this property. The TONCAS phenomenon can be described as follows: there is a reasonably small value n_0 such that if the desired property is satisfied for all subgraphs with n_0 or fewer vertices, then this propety is satisfied by the graph itself.

For planarity, as we have mentioned, $n_0 = 6$: if all subgraphs with 6 or fewer vertices are planar, this means that none of them is isomorphic to K_5 or to $K_{3,3}$ and thus, the whole graph is indeed planar. The similar bound $n_0 = 6$ holds for checking whether a lattice is distributive.

Describing TONCAS phenomenon in precise terms. Let us denote that by $P(n)$ the condition that all subgraphs with n or fewer vertices satisfy the desired property. Then, clearly, $P(n+1)$ implies $P(n)$.

In these terms, the fact that the whole graph – no matter how many vertices it has – satisfies the desired property means that the condition $P(n)$ holds for all n.

Thus, the TONCAS phenomenon means that there exists a value n_0 such that for each reasonable (= not abnormal) predicate P for which $P(n+1)$ implies $P(n)$, the condition $P(n_0)$ implies $\forall n\, P(n)$.

Caution. Of course, for each n_0, we can always find some artificial predicates $P(n)$ for which, for some graphs, we have $P(n_0)$ but $P(n)$ is not true for some $n > n_0$. This will happen, e.g., if the original desired graph property is that the graph has $\le n_0$ vertices. For this property, clearly $P(n_0)$ is true, but also clearly, $P(n_0 + 1)$ is not true for any graph with more than n_0 vertices.

So, to explain the TONCAS property, we cannot just ignore the words "reasonable" and "not abnormal", we need to formalize them.

3 How Can We Formalize What Is Not Abnormal

Let us use the experience of statistical physics. Many real-life phenomena are probabilistic. From the purely mathematical viewpoint, if we have, e.g., a Gaussian (normal) distribution on a real line, with 0 mean and standard deviation 1, then, since the probability density of the normal distribution is always positive, for every n – no matter how large it is – there is a positive probability that the random value will be larger than n. However, this is *not* how physicists reason; see, e.g., [2, 6].

For example, from the purely mathematical viewpoint, it is possible that, due to Brownian motion, a kettle placed on a cold stove will start boiling by itself – or that randomly moving molecules in a human body start moving in the same direction and the person will float into the air. A mathematician may say that this will happen if we wait a sufficiently long time – very long time, since the probabilities of these events are extremely small. However, this is *not* what a physicist will say. A physicist will simply claim that these events are *not* possible. In general, a physicist will say that if an event has a very low probability, then this even simply cannot happen.

How can we describe this physicists' reasoning in precise terms. Of course, we *cannot* simply fix a small number p_0 and claim that any event with probability $\le p_0$ is not possible. Indeed, in this case, for sufficiently large N – for which $2^{-N} \le p_0$ – we would come up with an awkward conclusion that it is impossible to flip a coin N times, since each of the 2^N possible sequences of heads and tails has the same probability $2^{-N} \le p_0$.

What we *can* conclude is that if we have a definable sequence of events $E_1 \supseteq E_2 \supseteq \ldots$ for which the probabilities $p(E_n)$ tend to 0, then at some point, the corresponding probability will be so small than the corresponding event E_n will simply not be possible. In other words, the class $\mathcal{R}$ of all *reasonable* (*not abnormal*) observations should satisfy the following property: if for some definable sequence $E_n \supseteq E_{n+1}$, we have $p(E_n) \to 0$, then there exists a value n_0 for which $\mathcal{R} \cap E_{n_0} = \emptyset$; see, e.g., [3, 4, 5].

What if we do not know probabilities? In statistical physics, we know the probabilities, but in graph-related situations, there is no natural way to assign probabilities. However, we can still use the above description if we take into account that there is a natural case when we can guarantee $p(E_n) \to 0$ for all possible probability distributions: namely, the case when the intersection $\bigcap\limits_n E_n$ of all the sets E_n is empty.

In this case, we arrive at the following description: the class $\mathcal{R}$ of all *reasonable* (*not abnormal*) observations should satisfy the following property: if for some definable sequence $E_n \supseteq E_{n+1}$, we have $\bigcap\limits_n E_n = \emptyset$, then there exists a value n_0 for which $\mathcal{R} \cap E_{n_0} = \emptyset$; see, e.g., [3, 4, 5].

4 Resulting Explanation of the TONCAS Phenomenon

Now, we are ready to explain the TONCAS phenomenon. We consider predicates $P(n)$ whose parameter n is a natural number. Let us call a predicate *monotonic* if for each n, $P(n+1)$ implies $P(n)$. We will prove that there exists a natural number n_0 such that if the monotonic predicate P is *not abnormal*, then $P(n_0)$ implies $\forall n\, P(n)$.

Indeed, let us consider the sets

$$E_n \stackrel{\text{def}}{=} \{P : P(1) \,\&\, \ldots \,\&\, P(n) \,\&\, \exists m\, \neg P(m)\}.$$

Clearly, here $E_n \supseteq E_{n+1}$ and $\bigcap\limits_n E_n = \emptyset$. Thus, by the above definition of not-abnormality, there exists an n_0 for which none of the not-abnormal monotonic predicates is contained in the set E_{n_0}. By the definition of the set E_{n_0}, this means that if we have $P(n_0)$ (and thus, due to monotonicity, $P(1) \,\&\, \ldots \,\&\, P(n_0)$), then we cannot have $\exists m\, \neg P(m)$ and thus, we have $\forall m\, P(m)$.

This is exactly the TONCAS phenomenon – which is, therefore, justified.

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