Egyptian Fractions Re-Revisited

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Abstract

Ancient Egyptians represented each fraction as a sum of unit fractions, i.e., fractions of the type $1/n$. In our previous papers, we explained that this representation makes perfect sense: e.g., it leads to an efficient way of dividing loaves of bread between people. However, one thing remained unclear: why, when representing fractions of the type $2/(2k+1)$, Egyptians did not use a natural representation $1/(2k+1)+1/(2k+1)$, but used a much more complicated representation instead. In this paper, we show that the need for such a complicated representation can be explained if we take into account that instead of cutting a rectangular-shaped loaf in one direction – as we considered earlier – we can simultaneously cut it in two orthogonal directions. For example, to cut a loaf into 6 pieces, we can cut in 2 pieces in one direction and in 3 pieces in another direction. Together, these cuts will divide the original loaf into $2\cdot 3=6$ pieces.

It is known that Egyptian fractions are an exciting topics for kids, helping them better understand fractions. In view of this fact, we plan to use our new explanation to further enhance this understanding.

1 Formulation of the Problem

Egyptian fractions: a brief reminder. Ancient Egyptians represented each fraction as a sum of unit fractions, i.e., fractions of the type $\dfrac{1}{n}$; see, e.g., [1, 2, 3]. According to the Rhind (Ahmes) Papyrus, the most detailed description of Egyptian mathematics, they used this representation, e.g., to divide loaves of bread between people.

Egyptian fractions are used in teaching. The Egyptian representation of fractions is an interesting and curious idea, very different from our usual representation of fractions – so different that it is often used in education to enhance the students' interest in studying fractions; see, e.g., [2, 4, 6, 7, 8, 9, 10].

But do they have any value beyond historical and pedagogical? The fact that the Egyptian fraction idea is so different from our modern mathematics may seem to indicate that this representation was not very efficient – otherwise, why was it forgotten and not followed?

Actually, the Egyptian fraction idea is efficient. As we have shown in [4, 5], the Egyptian fraction idea *is* efficient: e.g., in the bread division problem, it helps to minimize the number of cuts – and thus, minimize the effort to divide the bread.

Let us illustrate this efficiency on a simple example of an Egyptian representation of a fraction:
$$\frac{5}{6}=\frac{1}{2}+\frac{1}{3}.$$
The fraction $\frac{5}{6}$ corresponds to the task of dividing 5 loaves between 6 people. If we use a naive way of dividing, we divide each of the 5 loaves into 6 pieces. This requires 5 cuts per loaf, to the total of 25 cuts. (After the cutting, we give, to each person, 5 of the resulting pieces.)

The Egyptian-fraction way is:

- to divide $6\cdot\frac{1}{2}=3$ loaves into 2 pieces each (which requires 1 cut per loaf), and
- to divide the remaining $6\cdot\frac{1}{3}=2$ loaves into 3 pieces each (which requires 2 cuts per loaf).

As a result, we only need $3\cdot 1+2\cdot 2=7$ cuts – and one can show that this is indeed the smallest possible number of cuts needed for this bread division.

Remaining question. However, our argument did not explain why the Egyptians represented a fraction like $\frac{2}{2k+1}$ not in a natural way, as
$$\frac{2}{2k+1}=\frac{1}{2k+1}+\frac{1}{2k+1},$$
but in a much more complicated way, as
$$\frac{2}{2k+1}=\frac{1}{k+1}+\frac{1}{(k+1)\cdot(2k+1)}.$$
From the viewpoint of counting cuts, we do not seem to gain anything.

Indeed, the fraction $\dfrac{2}{2k+1}$ corresponding to dividing 2 loaves between $2k+1$ people – or, more generally, since

$$\frac{2}{2k+1} = \frac{2n}{(2k+1)\cdot n},$$

to dividing, for some natural number n, $2n$ loaves between $(2k+1)\cdot n$ people. To use the Egyptian representation, we need to take $n = k+1$. In this case, we divide $2(k+1)$ loaves between $(k+1)\cdot(2k+1)$ people.

For this problem, in the usual representation, we divide each of $2(k+1)$ loaves into $2k+1$ pieces – which requires $2k$ cuts per loaf, to the total of $2k\cdot 2(2k+1) = 4k^2+4k$ cuts.

If we use the Egyptian fraction representation, then:

- we divide $2k+1$ loaves into $k+1$ pieces each (which requires k cuts per loaf) and
- we divide the remaining loaf into $(k+1)(2k+1) = 2k^2+3k+1$ pieces – which requires $2k^2+3k$ cuts.

Thus, overall, we need $k\cdot(2k+1)+(2k^2+3k) = 4k^2+4k$ cuts – the same number as before. So why use this more complicated arrangement?

What we do in this paper. In this paper, we provide an explanation for this seemingly strange feature of Egyptian mathematics.

2 Our Explanation

Main idea behind our explanation. The main idea behind our explanation is related to the fact that in our previous paper, we considered 1-dimensional cuts, i.e., for a rectangular bread, cuts along one of the sides. In this case:

- if we want to divide a loaf into 2 pieces, we need 1 cut;
- if we want to divide a loaf into 3 pieces, we need 2 cuts;
- if we want to divide a loaf into 6 pieces, we need 5 cuts; and
- in general, if we want to divide a loaf into n pieces, we need $n-1$ cuts.

However, we can alternatively divide a rectangular loaf of bread in two orthogonal directions. For example, if we want to divide a piece of bread into 6 pieces, we can:

- divide it into 2 pieces in one direction (which requires 1 cut), and
- divide it into 3 pieces in another direction (which requires 2 cuts).

This way, we get $2 \cdot 3 = 6$ pieces with only $1 + 2 = 3$ cuts.

Let us show that this idea indeed explains why the Egyptians used a seemingly over-complicated representation of fractions of the type $\dfrac{2}{2k+1}$.

Comment. The possibility to cut in both directions does not mean that, e.g., a naive way of dividing 5 loaves between 6 people is better than the Egyptian way: even if we only use 3 cuts to divide each of the 5 loaves into 6 pieces, we will still need to make $3 \cdot 5 = 15$ cuts – which is more than twice as many as 7 cuts used by the Egyptian bread-cutters.

Actual explanation. In general, if we divide each of the $2(k+1)$ loaves into $2k+1$ pieces, we need $4k^2 + 4k$ cuts – as we have mentioned earlier.

If we use the seemingly weird Egyptian representation of the fraction $\dfrac{2}{2k+1}$, then:

- we need to divide $2k+1$ loaves into $k+1$ pieces each – which requires a total of $2k^2 + k$ cuts, and
- we need to divide one remaining loaf into $(k+1) \cdot (2k+1)$ pieces.

Good news is that to cut the remaining loaf, we can:

- cut it into $k+1$ pieces in one direction (which requires k cuts) and
- cut it into $2k+1$ pieces in another direction – which requires $2k$ cuts, to the total of $3k$ cuts.

Thus, in the Egyptian fraction approach, overall, we need $(2k^2 + k) + 3k = 2k^2 + 4k$ cuts.

This number of cuts is always smaller than $4k^2 + 4k$ cuts needed for the simpler representation. For large k, for which $k \ll k^2$, this practically twice smaller – which shows that the Egyptian representation of the fractions $\dfrac{2}{2k+1}$ is indeed much more efficient.

We plan to use this new idea in teaching fractions. As we have mentioned earlier, Egyptian fractions are an exciting topics for kids, helping them better understand fractions. In view of this fact, we plan to use our new explanation to further enhance this understanding.

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