Centroids Beyond Defuzzification

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Abstract In general, expert rules expressed by imprecise (fuzzy) words of natural language like "small" lead to imprecise (fuzzy) control recommendations. If we want to design an automatic controller, we need, based on these fuzzy recommendations, to generate a single control value. A procedure for such generation is known as defuzzification. The most widely used defuzzification procedure is centroid defuzzification, in which, as the desired control value, we use one of the coordinates of the center of mass ("centroid") of an appropriate 2-D set. A natural question is: what is the meaning of the second coordinate of this center of mass? In this paper, we show that this second coordinate describes the overall measure of fuzziness of the resulting recommendation.

1 Formulation of the Problem

Centroid defuzzification: a brief reminder. In fuzzy control (see, e.g., [1, 3, 4, 5, 6, 7]):

- we start with the expert rules formulated in terms in imprecise ("fuzzy") words from natural language, and

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- we end up with a strategy that, given the current values of the inputs, provides recommendations on what control value u to use.

This recommendation is also fuzzy: for each possible value u, the system provides a degree $\mu(u) \in [0,1]$ indicating to what extent this particular control value is reasonable for the given input.

- Such a fuzzy outcome is perfect if the main objective of the system is to advise a human controller.
- In many practical situations, however, we want this system to actually control.

In such situations, it is important to transform the fuzzy recommendation – as expressed by the function $\mu(u)$ (known as the *membership function*) – into a precise control value $\overline{u}$ that this system will apply. Such a transformation is known as *defuzzification*.

The most widely used defuzzification procedure is *centroid defuzzification*

$$\overline{u} = \frac{\int u \cdot \mu(u)\, du}{\int \mu(u)\, du}. \tag{1}$$

Geometric meaning of centroid defuzzification. The name for this defuzzification procedure comes from the fact that:

- if we take the subgraph of the function $\mu(u)$, i.e., the 2-D set

$$S \stackrel{\text{def}}{=} \{(u,y) : 0 \le y \le \mu(u)\},$$

- then the value (1) is actually the u-coordinate of this set's center of mass ("centroid") $(\overline{u},\overline{y})$.

Natural question. In fuzzy technique, we only use the u-coordinate of the center of mass. A natural question is: is there a fuzzy-related meaning of the y-coordinate $\overline{y}$?

What we do in this paper. In this paper, we propose such a meaning.

Comment. This paper follows our preliminary results published in [2].

2 Fuzzy-Related Meaning of the "Other" Component of the Centroid

Mathematical formula for the y-component. In general, the y-component of the center of mass of a 2-D body S has the form

$$\overline{y} = \frac{\int_S y\, du\, dy}{\int_S du\, dy}.$$

The denominator is the same as for the u-component: it is equal to $\int \mu(u)\,du$. The numerator can also be easily computed as

$$\int_S y\,du\,dy = \int_u du \cdot \int_0^{\mu(u)} y\,dy = \frac{1}{2} \cdot \int_u \mu^2(u)\,du.$$

Thus, we have

$$\overline{y} = \frac{1}{2} \cdot \frac{\int \mu^2(u)\,du}{\int \mu(u)\,du}. \tag{2}$$

First meaning of this formula. The u-component of the centroid is the weighted average value of u, with weights proportional to $\mu(u)$. Similarly, the expression (2) is the weighted average value of $\mu(u)$.

Each value $\mu(u)$ is the degree of fuzziness of the system's recommendation about the control value u. Thus, the value $\overline{y}$ can be viewed with the weighted average value of the degree of fuzziness.

Let us show that this interpretation makes some sense.

Proposition 1.

- *The value $\overline{y}$ is always between 0 and 1/2.*
- *For a measurable function $\mu(u)$, the value $\overline{y}$ is equal to 1/2 if and only if the value $\mu(u)$ is almost everywhere equal either to 0 or to 1.*

Comments.

- In other words, if we ignore sets of measure 0, the value $\overline{y}$ is equal to 1/2 if and only if the corresponding fuzzy set is actually crisp. For all non-crisp fuzzy sets, we have $\overline{y} < 1/2$.
- For a triangular membership function, one can check that we always have $\overline{y} = 1/3$. For trapezoid membership functions, $\overline{y}$ can take any possible value between 1/3 and 1/2: the larger the value-1 part, the larger $\overline{y}$.

Proof of Proposition 1. Since $\mu(u) \in [0,1]$, we always have $\mu^2(u) \le \mu(u)$, thus $\int \mu^2(u)\,du \le \int \mu(u)\,du$, hence

$$\frac{\int \mu^2(u)\,du}{\int \mu(u)\,du} \le 1$$

and $\overline{y} \le 1/2$.

Vice versa, if $\overline{y} = 1/2$, this means that

$$\frac{\int \mu^2(u)\,du}{\int \mu(u)\,du} = 1.$$

Multiplying both sides of this equality by the denominator, we conclude that $\int \mu^2(u)\,du = \int \mu(u)\,du$, i.e., that $\int \left(\mu(u) - \mu^2(u)\right)\,du = 0$. As we have mentioned, the difference $\mu(u)-\mu^2(u)$ is always non-negative. Since its integral is 0, this means that this difference is almost always equal to 0 – and the equality

$$\mu(u)-\mu^2(u)=0$$

means that either $\mu(u)=0$ or $\mu(u)=1$.

The proposition is proven.

A version of the first meaning. In general, in the fuzzy case, we have different values of the degree of confidence $\mu(u)$ for different possible control values u. A natural way to find the "average" degree of fuzziness is to find a single degree μ_0 which best represents all these values. This is natural to interpret as requiring that the mean square difference weighted by $\mu(u)$ – i.e., the value

$$\int (\mu(u)-\mu_0)^2\cdot\mu(u)\,du$$

attains its smallest possible value. Differentiating the minimized expression with respect to μ_0 and equating the derivative to 0, we conclude that

$$\int 2(\mu_0-\mu(u))\cdot\mu(u)\,du=0,$$

hence

$$\mu_0=\frac{\int \mu^2(u)\,du}{\int \mu(u)\,du}.$$

Thus, $y_0=(1/2)\cdot\mu_0$.

Second meaning. It is known that from the mathematical viewpoint, membership functions $\mu(u)$ and probability density functions $\rho(u)$ differ by their normalization:

- for a membership function $\mu(u)$, we require that $\max\limits_u \mu(u)=1$, while
- for a probability density function $\rho(u)$, we require that $\int \rho(u)\,du=1$.

For every non-negative function $f(u)$, we can divide it by an appropriate constant c and get an example of either a membership function or a probability density function:

- if we divide $f(u)$ by $c=\max\limits_v f(v)$, then we get a membership function
$$\mu(u)=\frac{f(u)}{\max\limits_v f(v)};$$
- if we divide $f(u)$ by $c=\int \mu(v)\,dv$, we get a probability density function
$$\rho(u)=\frac{f(u)}{\int f(v)\,dv}.$$

In particular, for each membership function $\mu(u)$, we can construct the corresponding probability density function

$$\rho(u) = \frac{\mu(u)}{\int \mu(v)\,dv}.$$

In terms of this expression $\rho(u)$, the formulas for both components of the center of mass take a simplified form:

- the result $\overline{u}$ of centroid defuzzification takes the form $\overline{u} = \int u \cdot \rho(u)\,du$, i.e., is simply equal to the expected value of control under this probability distribution;
- similarly, the value $\mu_0 = 2\overline{y}$ takes the form $\mu_0 = \int \mu(u) \cdot \rho(u)\,du$, i.e., is equal to the expected value of the membership function.

It should be mentioned that the formula $\int \mu(u) \cdot \rho(u)\,du$ was first proposed by Zadeh himself to describe the probability of the fuzzy event characterized by the membership function $\mu(u)$. Since this membership function characterizes to what extent a control value u is reasonable, the value μ_0 thus describes the probability that a control value selected by fuzzy control will be reasonable.

This interpretation is in good accordance with Proposition 1:

- if we are absolutely confident in our recommendations, i.e., if $\mu(u)$ is a crisp set, then this probability μ_0 is equal to 1 – and thus, $\overline{y} = (1/2) \cdot \mu_0$ is equal to 1/2;
- on the other hand, if we are not confident in our recommendations, then the probability μ_0 is smaller than 1 and thus, its half $\overline{y}$ is smaller than 1/2.

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