Decision Making Under Interval Uncertainty Revisited

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Abstract

In many real-life situations, we do not know the exact values of the expected gain corresponding to different possible actions, we only have lower and upper bounds on these gains – i.e., in effect, intervals of possible gain values. How can we made decisions under such interval uncertainty? In this paper, we show that natural requirements lead to a 2-parametric family of possible decision-making strategies.

1 Formulation of the Problem

Decision making: a brief reminder. In many real-life situations, we need to select an appropriate action. In economics, a reasonable idea is to select an action that leads to the largest values of the expected gain; see, e.g., [1, 3, 4, 5, 6]. This way, if we repeatedly make such a selection, then, due to the law of large numbers, we will get the largest possible gain.

Need to take uncertainty into account. In practice, we often we often cannot predict the exact consequence of each possible action. As a result, for each action a, instead of the exact value u_a of the expected gain, we only know the *interval* $[\underline{u}_a, \overline{u}_a]$ of possible gain values.

Comment. It is often convenient to represent this interval in the equivalent form, as $[\widetilde{u}_a - \Delta_a, \widetilde{u}_a + \Delta_a]$, where

$$\widetilde{u}_a = \frac{\underline{u}_a + \overline{u}_a}{2} \text{ and } \Delta_a = \frac{\overline{u}_a - \underline{u}_a}{2}.$$

How to make decisions under interval uncertainty. How can we make decisions under such interval uncertainty? In other words, when can we decide that Action 1 is better than Action 2? In general, we have three possible cases.

Sometimes, we can guarantee that Action 1 is better than Action 2. This happens if every value u_1 from the interval $[\underline{u}_1, \overline{u}_1]$ is larger than or equal to every value u_2 from the interval $[\underline{u}_2, \overline{u}_2]$. One can easily check that this is equivalent to requiring that the smallest possible value $\underline{u}_1$ from the interval $[\underline{u}_1, \overline{u}_1]$ is larger than or equal to the largest possible value $\overline{u}_2$ from the interval $[\underline{u}_2, \overline{u}_2]$, i.e., that:

$$\overline{u}_1 \leq \underline{u}_2.$$

Sometimes, we can guarantee that Action 2 is better than Action 1, i.e., that every value u_2 from the interval $[\underline{u}_2, \overline{u}_2]$ is larger than or equal to every value u_1 from the interval $[\underline{u}_1, \overline{u}_1]$. Similarly to the previous case, we can conclude that this condition is equivalent to:

$$\overline{u}_1 \leq \overline{u}_2.$$

In all other cases, i.e., when

$$\underline{u}_2 < \overline{u}_1 \text{ and } \underline{u}_1 < \overline{u}_2,$$

we cannot make a guaranteed conclusion: in such cases, it can be that Action 1 is better, and it can also be that Action 2 is better.

So which action should we select? In situations in which we can guarantee that one of the actions if better, this better action is the one we should select. But what if we are in the situation when no such guarantee is possible? Which action should we then recommend?

This is a question that we consider in this paper.

Comment. It is not necessary to provide recommendation for *all* the cases, but we would like to be able to provide recommendation for at least *some* of the cases.

2 Analysis of the Problem

What do we want. We want to be able, for some intervals $[\underline{u}_1, \overline{u}_1]$ and $[\underline{u}_2, \overline{u}_2]$, to say that the second interval is better (or of the same quality). We will denote this relation by the usual inequality sign:

$$[\underline{u}_1, \overline{u}_1] \leq [\underline{u}_2, \overline{u}_2].$$

What are the natural requirements on this relation?

First natural requirement: transitivity. If Action 2 is better than (or of the same quality as) Action 1, and Action 3 is better than (or of the same quality as) Action 2, then we should be able to conclude that Action 3 is better than (or of the same quality as) Action 1, i.e., that the relation $\leq$ on the class of all intervals should be transitive:

$$\text{if } [\underline{u}_1, \overline{u}_1] \leq [\underline{u}_2, \overline{u}_2] \text{ and } [\underline{u}_2, \overline{u}_2] \leq [\underline{u}_3, \overline{u}_3], \text{ then } [\underline{u}_1, \overline{u}_1] \leq [\underline{u}_3, \overline{u}_3]. \quad (1)$$

Second natural requirement: reflexivity. Each interval has the same quality as itself. So, for every interval $[\underline{u}, \overline{u}]$, we should have:

$$[\underline{u}, \overline{u}] \leq [\underline{u}, \overline{u}]. \tag{2}$$

Third natural requirement: consistency with common sense. It is reasonable to require that if Action 2 is guaranteed to be better than Action 1, then we will still select Action 2:

$$\text{if } \overline{u}_1 \leq \underline{u}_2, \text{ then } [\underline{u}_1, \overline{u}_1] \leq [\underline{u}_2, \overline{u}_2]. \tag{3}$$

Fourth natural requirement: scale-invariance. If we multiply all the gains by the same positive constant $c > 0$, then whichever gain was larger remains larger, and whichever gain was smaller remains smaller. This multiplication corresponds, e.g., to switching from the original currency to the one which is c times smaller: the mere change of currency should not change which action is better. It is therefore reasonable to require that a similar change of currency should not affect decision making under uncertainty either, i.e., that:

$$\text{if } [\underline{u}_1, \overline{u}_1] \leq [\underline{u}_2, \overline{u}_2] \text{ then } [c \cdot \underline{u}_1, c \cdot \overline{u}_1] \leq [c \cdot \underline{u}_2, c \cdot \overline{u}_2]. \tag{4}$$

Fifth natural requirement: additivity. If we add the same amount to the two gains, this will not change which gain is larger. Similarly, if we add the same interval-valued gain $[\underline{c}, \overline{c}]$ to the gains of both actions, this should not change which action was better.

If we have two independent situations, in one of which the gain can be anything from $\underline{u}_i$ to $\overline{u}_i$ and in the second one anything from $\underline{c}$ to $\overline{c}$, then the smallest possible value of the overall gain is when both gains are the smallest, i.e., when we have $\underline{u}_i + \underline{c}$, and the largest possible value of the overall gain is when both gains are the largest, i.e., when we have $\overline{u}_i + \overline{c}$.

Thus, the above requirement takes the following form:

$$[\underline{u}_1, \overline{u}_1] \leq [\underline{u}_2, \overline{u}_2] \text{ if and only if } [\underline{u}_1 + \underline{c}, \overline{u}_1 + \overline{c}] \leq [\underline{u}_2 + \underline{c}, \overline{u}_2 + \overline{c}]. \tag{5}$$

Final natural requirement: closeness. When the values of $\underline{u}$ and $\overline{u}$ are close, the corresponding alternatives are practically indistinguishable. Thus, it is reasonable to require that if we have two sequences of intervals $\left[\underline{u}_1^{(n)}, \overline{u}_1^{(n)}\right]$ and $\left[\underline{u}_2^{(n)}, \overline{u}_2^{(n)}\right]$ for which $\left[\underline{u}_1^{(n)}, \overline{u}_1^{(n)}\right] \leq \left[\underline{u}_2^{(n)}, \overline{u}_2^{(n)}\right]$, and endpoints of both intervals tends to some limits, then, since the limit intervals are indistinguishable from these one for sufficiently large n, we should expect the same relation $\le$ for the limit intervals as well:

$$\text{if } \left[\underline{u}_1^{(n)}, \overline{u}(n)_1\right] \le \left[\underline{u}_2^{(n)}, \overline{u}_2^{(n)}\right] \text{ for all } n, \text{ and } \underline{u}_i^{(n)} \to \underline{u}_i \text{ and } \overline{u}_i^{(n)} \to \overline{u}_i,$$

$$\text{then } [\underline{u}_1, \overline{u}_1] \le [\underline{u}_2, \overline{u}_2]. \tag{6}$$

Now, we are ready to formulate our main result.

3 Definitions and the Main Result

Definition.

- *We say that a binary relation $\le$ on the set of all intervals is* transitive *if it satisfies the condition* (1)*.*
- *We say that a binary relation $\le$ on the set of all intervals is* reflexive *if it satisfies the condition* (2)*.*
- *We say that a binary relation $\le$ on the set of all intervals is* consistent with common sense *if it satisfies the condition* (3)*.*
- *We say that a binary relation $\le$ on the set of all intervals is* scale-invariant *if it satisfies the condition* (4)*.*
- *We say that a binary relation $\le$ on the set of all intervals is* additive *if it satisfies the condition* (5)*.*
- *We say that a binary relation $\le$ on the set of all intervals is* closed *if it satisfies the condition* (6)*.*

Proposition. *For a binary relation $\le$ on the set of all intervals, the following two conditions are equivalent to each other:*

- *the relation is transitive, reflexive, consistent with common sense, scale-invariant, additive, and closed;*
- *for some values α_- and α_+ for which $-1 \le \alpha_- \le \alpha_+ \le 1$, the relation $\le$ has the following form: $[\widetilde{u}_1 - \Delta_1, \widetilde{u}_1 + \Delta_1] \le [\widetilde{u}_2 - \Delta_2, \widetilde{u}_2 + \Delta_2]$ if and only if either*

  $$\Delta_1 \le \Delta_2 \text{ and } \widetilde{u}_1 + \alpha_- \cdot \Delta_1 \le \widetilde{u}_2 + \alpha_- \cdot \Delta_2$$

  or

  $$\Delta_2 \le \Delta_1 \text{ and } \widetilde{u}_1 + \alpha_+ \cdot \Delta_1 \le \widetilde{u}_2 + \alpha_+ \cdot \Delta_2.$$

Comments.

- The only case when we have a linear order, i.e., when for every two intervals $[\underline{u}_1, \overline{u}_1]$ and $[\underline{u}_2, \overline{u}_2]$, we have either $[\underline{u}_1, \overline{u}_1] \le [\underline{u}_2, \overline{u}_2]$ or $[\underline{u}_2, \overline{u}_2] \le [\underline{u}_1, \overline{u}_1]$, is when $\alpha_- = \alpha^+$. In this case, we get the known Hurwicz criterion for decision making; see, e.g., [2, 3, 4].
- Relations described in the Proposition were first considered in [7, 8], but with an additional requirement that $[\underline{u}_1, \overline{u}_1] \le [\underline{u}_2, \overline{u}_2]$ implies $\widetilde{u}_1 \le \widetilde{u}_2$. This requirement is not always satisfied: e.g., for $\alpha_- = \alpha_+ = 1$, when

  $$[\underline{u}_1, \overline{u}_1] \le [\underline{u}_2, \overline{u}_2] \text{ if and only if } \overline{u}_1 \le \overline{u}_2,$$

  we have $[1, 1] \le [-2, 2]$ but for the midpoints $\widetilde{u}_i$ of these intervals, the opposite inequality is true: $1 > 0$.

4 Proof

1°. It is straightforward to prove that every relation of the above form indeed satisfies conditions (1) through (6). So, to complete the proof, it is sufficient to prove that if a relation satisfies the conditions (1)–(6), then it has the desired form.

2°. Let us first analyze how the interval $[-1, 1]$ compares with different real values u (i.e., with degenerate intervals $[u, u]$).

2.1°. Due to consistency with common sense, we have $u \le [-1, 1]$ when $u \le -1$. Let us denote $\alpha_- \stackrel{\text{def}}{=} \sup\{u : u \le [-1, 1]\}$. This value is a limit of values for which $u \le [-1, 1]$, thus due to closeness, $\alpha_- \le [-1, 1]$.

By transitivity, if $u \le \alpha_-$, then we also have $u \le [-1, 1]$. By definition of α_-, if $u > \alpha_-$, then we cannot have $u \le [-1, 1]$. Thus, we have

$$u \le [-1, 1] \text{ if and only if } u \le \alpha_-. \tag{7}$$

2.2°. Similarly, due to consistency with common sense, we have $[-1, 1] \le u$ when $1 \le u$. Let us denote $\alpha_+ \stackrel{\text{def}}{=} \inf\{u : [-1, 1] \le u\}$. This value is a limit of values for which $[-1, 1] \le u$, thus due to closeness, $[-1, 1] \le \alpha_+$.

By transitivity, if $\alpha_+ \le u$, then we also have $[-1, 1] \le u$. By definition of α_+, if $u < \alpha_+$, then we cannot have $[-1, 1] \le u$. Thus, we have

$$[-1, 1] \le u \text{ if and only if } \alpha_+ \le u. \tag{8}$$

3°. Let us now compare two general intervals

$$[\widetilde{u}_1 - \Delta_1, \widetilde{u}_1 + \Delta_1] \text{ and } [\widetilde{u}_2 - \Delta_2, \widetilde{u}_2 + \Delta_2].$$

There are three possible cases that we will consider one by one: when $\Delta_1 = \Delta_2$, when $\Delta_1 < \Delta_2$, and when $\Delta_2 < \Delta_1$.

3.1°. When $\Delta_1 = \Delta_2$, then for $[\underline{c}, \overline{c}] = [-\Delta_1, \Delta_1] = [-\Delta_2, \Delta_2]$, additivity implies that

$$[\widetilde{u}_1 - \Delta_1, \widetilde{u}_1 + \Delta_1] \le [\widetilde{u}_2 - \Delta_2, \widetilde{u}_2 + \Delta_2] \text{ if and only if } \widetilde{u}_1 \le \widetilde{u}_2.$$

So, in this case, the proposition is proven.

3.2°. Let us now consider the case when $\Delta_1 < \Delta_2$. Then, for

$$[\underline{c}, \overline{c}] = [\widetilde{u}_2 - \Delta_1, \widetilde{u}_2 + \Delta_1],$$

additivity implies that

$$[\widetilde{u}_1 - \Delta_1, \widetilde{u}_1 + \Delta_1] \le [\widetilde{u}_2 - \Delta_2, \widetilde{u}_2 + \Delta_2] \text{ if and only if}$$

$$\widetilde{u}_1 - \widetilde{u}_2 \le [-(\Delta_2 - \Delta_1), \Delta_2 - \Delta_1]. \tag{9}$$

By applying scale-invariance with $c = \Delta_2 - \Delta_1 > 0$, we conclude that

$$\widetilde{u}_1 - \widetilde{u}_2 \le [-(\Delta_2 - \Delta_1), \Delta_2 - \Delta_1] \text{ if and only if } \frac{\widetilde{u}_1 - \widetilde{u}_2}{\Delta_2 - \Delta_1} \le [-1, 1]. \tag{10}$$

Due to (7), this inequality is, in its turn, equivalent to

$$\frac{\widetilde{u}_1 - \widetilde{u}_2}{\Delta_2 - \Delta_1} \le \alpha_-. \tag{11}$$

Multiplying both sides of this inequality by the positive number $\Delta_2 - \Delta_1$, we get an equivalent inequality

$$\widetilde{u}_1 - \widetilde{u}_2 \le \alpha_- \cdot \Delta_2 - \alpha_- \cdot \Delta_1, \tag{12}$$

i.e., equivalently,

$$\widetilde{u}_1 + \alpha_- \cdot \Delta_1 \le \widetilde{u}_2 + \alpha_- \cdot \Delta_2. \tag{13}$$

Thus, from (9)–(13), we conclude that here, indeed:

$$[\widetilde{u}_1 - \Delta_1, \widetilde{u}_1 + \Delta_1] \le [\widetilde{u}_2 - \Delta_2, \widetilde{u}_2 + \Delta_2] \text{ if and only if}$$

$$\widetilde{u}_1 + \alpha_- \cdot \Delta_1 \le \widetilde{u}_2 + \alpha_- \cdot \Delta_2. \tag{14}$$

3.3°. To complete the proof, we need to consider the case when $\Delta_2 < \Delta_1$. Then, for $[\underline{c}, \overline{c}] = [\widetilde{u}_1 - \Delta_2, \widetilde{u}_1 + \Delta_2]$, additivity implies that

$$[\widetilde{u}_1 - \Delta_1, \widetilde{u}_1 + \Delta_1] \le [\widetilde{u}_2 - \Delta_2, \widetilde{u}_2 + \Delta_2] \text{ if and only if}$$

$$[-(\Delta_1 - \Delta_2), \Delta_1 - \Delta_2] \le \widetilde{u}_2 - \widetilde{u}_1. \tag{15}$$

By applying scale-invariance with $c = \Delta_1 - \Delta_2 > 0$, we conclude that

$$[-(\Delta_1 - \Delta_2), \Delta_1 - \Delta_2] \le \widetilde{u}_2 - \widetilde{u}_1 \text{ if and only if } [-1, 1] \le \frac{\widetilde{u}_2 - \widetilde{u}_1}{\Delta_1 - \Delta_2}. \tag{16}$$

Due to (8), this inequality is, in its turn, equivalent to

$$\alpha_+ \le \frac{\widetilde{u}_2 - \widetilde{u}_1}{\Delta_1 - \Delta_2}. \tag{17}$$

Multiplying both sides of this inequality by the positive number $\Delta_1 - \Delta_2$, we get an equivalent inequality

$$\alpha_+ \cdot \Delta_1 - \alpha_+ \cdot \Delta_2 \le \widetilde{u}_2 - \widetilde{u}_1, \tag{18}$$

i.e., equivalently,

$$\widetilde{u}_1 + \alpha_+ \cdot \Delta_1 \le \widetilde{u}_2 + \alpha_+ \cdot \Delta_2. \tag{19}$$

Thus, from (15)–(19), we conclude that here, indeed:

$$[\widetilde{u}_1 - \Delta_1, \widetilde{u}_1 + \Delta_1] \le [\widetilde{u}_2 - \Delta_2, \widetilde{u}_2 + \Delta_2] \text{ if and only if}$$

$$\widetilde{u}_1 + \alpha_+ \cdot \Delta_1 \le \widetilde{u}_2 + \alpha_+ \cdot \Delta_2. \tag{20}$$

So, in all three cases, the proposition is proven.

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