A Fully Lexicographic Extension of Min or Max Operation Cannot Be Associative

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Abstract

In many applications of fuzzy logic, to estimate the degree of confidence in a statement $A \,\&\, B$, we take the minimum $\min(a, b)$ of the expert's degrees of confidence in the two statements A and B. When $a < b$, then an increase in b does not change this estimate, while from the commonsense viewpoint, our degree of confidence in $A \,\&\, B$ should increase. To take this commonsense idea into account, Ildar Batyrshin and colleagues proposed to extend the original order in the interval $[0, 1]$ to a lexicographic order on a larger set. This idea works for expressions of the type $A \,\&\, B$, so maybe we can extend it to more general expressions? In this paper, we show that such an extension, while theoretically possible, would violate another commonsense requirement – associativity of the "and"-operation. A similar negative result is proven for lexicographic extensions of the maximum operation – that estimates the expert's degree of confidence in a statement $A \vee B$.

1 Formulation of the Problem

Min and max operations: reminder. In fuzzy logic (see, e.g., [5, 8, 9, 10, 11, 12, 15]), the expert's degree of confidence in a statement is described by a number from the interval $[0, 1]$.

Often, we know the expert's degree of certainty a and b of statements A and B, and, based on these two values, we need estimate the expert's degree of confidence in a composite statement $A \,\&\, B$. The corresponding estimate can be denoted by $a \,\&\, b$. In many applications, we have $a \,\&\, b = \min(a, b)$.

Similarly, as an estimate $a \vee b$ for the expert's degree of confidence in a composite statement $A \vee B$, often, the max operation $a \vee b = \max(a, b)$ is used.

Need to describe a subtle difference. According to the usual min-operation, $a \,\&\, a = a$ and $a \,\&\, 1 = a$, so the value $a \,\&\, b$ remains the same when $b = a$ and when $b = 1$. However, intuitively, if we increase our degree of confidence in a statement B, the degree of confidence in a composite statement $A \,\&\, B$ should increase. Thus, we expect that $a \,\&\, a < a \,\&\, 1$. In other words, instead of a single common value $a \in [0, 1]$, we should have different values $a \,\&\, b$ corresponding to different $b \in [a, 1]$.

In other words, we need to extend the interval $[0, 1]$ to a larger set, and extend the original order to the new set, so that:

- what was smaller remains smaller, but
- what was equal may not remain equal anymore.

How can we describe this subtle difference. How can we compare expressions $a \,\&\, b$? Since the "and"-operation is naturally commutative, we can, without losing generality, order a and b in increasing order, i.e., we can always assume that $a \le b$.

How can we compare expressions $a_1 \,\&\, b_1$ and $a_2 \,\&\, b_2$ in which $a_1 \le b_1$ and $a_2 \le b_2$? If $a_1 < a_2$, then for the min-operation, we have

$$a_1 \,\&\, b_1 = a_1 < a_2 = a_2 \,\&\, b_2.$$

Since we want to retain the previous order, we thus conclude that $a_1 \,\&\, b_1 < a_2 \,\&\, b_2$ in the desired extension as well.

If $a_2 < a_1$, then similarly, we should have $a_2 \,\&\, b_2 < a_1 \,\&\, b_1$.

What if $a_1 = a_2$? In this case, for the min-operation, we get equality, but this is exactly the equality that want to clarify, so we say that $a_1 \,\&\, b_1 < a_2 \,\&\, b_2$ if $a_1 = a_2$ and $b_1 < b_2$.

This order on expressions $a \,\&\, b$ can be naturally extended to values $a \in [0, 1]$, since each such value can be described as $a \,\&\, 1$.

So, we arrive at the following *lexicographic order:* when $a_1 \le b_1$ and $a_2 \le b_2$, then $a_1 \,\&\, b_1 \le a_2 \,\&\, b_2$ if and only if:

- either $a_1 < a_2$,
- or $a_1 = a_2$ and $b_1 \le b_2$.

Such an order was first proposed in [1, 2, 3, 4, 14]. It was successfully used in applications to geosciences; see, e.g., [14].

Natural question. The idea of a lexicographic order works well for expressions of the type $a \,\&\, b$. Can we extend this idea to more general expressions?

In this paper, we show that while such an extension is possible, it is not what we look for: e.g., the corresponding operation will *not* be associative – while we want associativity $a \,\&\, (b \,\&\, c) = (a \,\&\, b) \,\&\, c$, since, from the common sense viewpoint, $A \,\&\, (B \,\&\, C)$ means exactly the same as $(A \,\&\, B) \,\&\, C$: that all three statement A, B, and C are true.

A similar result is also proven for a similar lexicographic extension of the max-operation.

2 Main Result: Case of Min Operation

Definition 1. *Let $(S, \le)$ be a partially ordered set with the largest element 1 that contains two elements a and b for which $a < b < 1$. Let & be a commutative operation on the set S for which $a \,\&\, 1 = a$ for all a. We say that the order $\le$ is* lexicographic *if for all $a_1 \le b_1$ and $a_2 \le b_2$, we have $a_1 \,\&\, b_1 \le a_2 \,\&\, b_2$ if and only if:*

- *either $a_1 < a_2$,*
- *or $a_1 = a_2$ and $b_1 \le b_2$.*

Proposition 1. *When the order is lexicographic, the operation & is not associative.*

Proof. Let us consider the elements $a < b < 1$ whose existence is guaranteed by the definition of lexicographic order. Then, by this definition, for $a_1 = a_2 = a$, $b_1 = b$, and $b_2 = 1$, we get

$$a \,\&\, b < a \,\&\, 1. \quad (1)$$

From $a < b$ and from the fact that $a \,\&\, 1 = a$, we conclude that

$$a \,\&\, 1 = a < b. \quad (2)$$

Now, for $a_1 = a \,\&\, b$, $a_2 = a \,\&\, 1$, $b_1 = 1$, and $b_2 = b$:

- we have $a_1 \le b_1$ – since 1 is the largest element,
- we have $a_2 \le b_2$ – by formula (2), and
- we have $a_1 < a_2$ – by formula (1).

So, since the order is lexicographic, we can conclude that $a_1 \,\&\, b_1 < a_2 \,\&\, b_2$, i.e., that

$$(a \,\&\, b) \,\&\, 1 < (a \,\&\, 1) \,\&\, b, \quad (3)$$

while by associativity and commutativity, we would have $(a \,\&\, b) \,\&\, 1 = (a \,\&\, 1) \,\&\, b$. Thus, the operation & is not associative.

The proposition is proven.

3 Main Result: Case of Max Operation

Definition 2. *Let $(S, \le)$ be a partially ordered set with the smallest element 0 that contains two elements a and b for which $0 < a < b$. Let $\vee$ be a commutative operation on the set S for which $a \vee 0 = a$ for all a. We say that the order $\le$ is* lexicographic *if for all $a_1 \le b_1$ and $a_2 \le b_2$, we have $a_1 \vee b_1 \le a_2 \vee b_2$ if and only if:*

- *either $b_1 < b_2$,*

- *or* $b_1 = b_2$ *and* $a_1 \le a_2$.

Proposition 2. *When the order is lexicographic, the operation* $\vee$ *is not associative.*

Proof. Let us consider the elements $0 < a < b$ whose existence is guaranteed by the definition of lexicographic order. Then, by this definition, for $a_1 = 0 < a_2 = a$, and $b_1 = b_2 = b$, we get

$$0 \vee b < a \vee b. \tag{4}$$

From $a < b$ and from the fact that $0 \vee b = b$, we conclude that

$$a < 0 \vee b = b. \tag{5}$$

Now, for $a_1 = a$, $a_2 = 0$, $b_1 = 0 \vee b$, and $b_2 = a \vee b$:

- we have $a_1 \le b_1$ – by formula (5),
- we have $a_2 \le b_2$ – since 0 is the smallest element, and
- we have $b_1 < b_2$ – by formula (4).

So, since the order is lexicographic, we can conclude that $a_1 \vee b_1 < a_2 \vee b_2$, i.e., that

$$a \vee (0 \vee b) < 0 \vee (a \vee b), \tag{6}$$

while by associativity and commutativity, we would have $a \vee (0 \vee b) = 0 \vee (a \vee b)$. Thus, the operation $\vee$ is not associative.

The proposition is proven.

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