A Fully Lexicographic Extension of Min or Max Operation Cannot Be Associative

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Abstract
In many applications of fuzzy logic, to estimate the degree of confidence in a statement $A \& B$, we take the minimum $\min(a, b)$ of the expert’s degrees of confidence in the two statements $A$ and $B$. When $a < b$, then an increase in $b$ does not change this estimate, while from the commonsense viewpoint, our degree of confidence in $A \& B$ should increase. To take this commonsense idea into account, Ildar Batyrshin and colleagues proposed to extend the original order in the interval $[0, 1]$ to a lexicographic order on a larger set. This idea works for expressions of the type $A \& B$, so maybe we can extend it to more general expressions? In this paper, we show that such an extension, while theoretically possible, would violate another commonsense requirement – associativity of the “and”-operation. A similar negative result is proven for lexicographic extensions of the maximum operation – that estimates the expert’s degree of confidence in a statement $A \lor B$.

1 Formulation of the Problem

Min and max operations: reminder. In fuzzy logic (see, e.g., [5, 8, 9, 10, 11, 12, 15]), the expert’s degree of confidence in a statement is described by a number from the interval $[0, 1]$.

Often, we know the expert’s degree of certainty $a$ and $b$ of statements $A$ and $B$, and, based on these two values, we need estimate the expert’s degree of confidence in a composite statement $A \& B$. The corresponding estimate can be denoted by $a \& b$. In many applications, we have $a \& b = \min(a, b)$.

Similarly, as an estimate $a \lor b$ for the expert’s degree of confidence in a composite statement $A \lor B$, often, the max operation $a \lor b = \max(a, b)$ is used.

Need to describe a subtle difference. According to the usual min-operation, $a \& a = a$ and $a \& 1 = a$, so the value $a \& b$ remains the same when $b = a$ and
when $b = 1$. However, intuitively, if we increase our degree of confidence in a statement $B$, the degree of confidence in a composite statement $A \& B$ should increase. Thus, we expect that $a \& a < a \& 1$. In other words, instead of a single common value $a \in [0, 1]$, we should have different values $a \& b$ corresponding to different $b \in [a, 1]$. In other words, we need to extend the interval $[0, 1]$ to a larger set, and extend the original order to the new set, so that:

- what was smaller remains smaller, but
- what was equal may not remain equal anymore.

**How can we describe this subtle difference.** How can we compare expressions $a \& b$? Since the “and”-operation is naturally commutative, we can, without losing generality, order $a$ and $b$ in increasing order, i.e., we can always assume that $a \leq b$.

How can we compare expressions $a_1 \& b_1$ and $a_2 \& b_2$ in which $a_1 \leq b_1$ and $a_2 \leq b_2$? If $a_1 < a_2$, then for the min-operation, we have

$$a_1 \& b_1 = a_1 < a_2 = a_2 \& b_2.$$  

Since we want to retain the previous order, we thus conclude that $a_1 \& b_1 < a_2 \& b_2$ in the desired extension as well.

If $a_2 < a_1$, then similarly, we should have $a_2 \& b_2 < a_1 \& b_1$.

What if $a_1 = a_2$? In this case, for the min-operation, we get equality, but this is exactly the equality that want to clarify, so we say that $a_1 \& b_1 < a_2 \& b_2$ if $a_1 = a_2$ and $b_1 < b_2$.

This order on expressions $a \& b$ can be naturally extended to values $a \in [0, 1]$, since each such value can be described as $a \& 1$.

So, we arrive at the following lexicographic order: when $a_1 \leq b_1$ and $a_2 \leq b_2$, then $a_1 \& b_1 \leq a_2 \& b_2$ if and only if:

- either $a_1 < a_2$,
- or $a_1 = a_2$ and $b_1 \leq b_2$.

Such an order was first proposed in $[1, 2, 3, 4, 14]$. It was successfully used in applications to geosciences; see, e.g., $[14]$.

**Natural question.** The idea of a lexicographic order works well for expressions of the type $a \& b$. Can we extend this idea to more general expressions?

In this paper, we show that while such an extension is possible, it is not what we look for: e.g., the corresponding operation will not be associative – while we want associativity $a \& (b \& c) = (a \& b) \& c$, since, from the common sense viewpoint, $A \& (B \& C)$ means exactly the same as $(A \& B) \& C$: that all three statement $A$, $B$, and $C$ are true.

A similar result is also proven for a similar lexicographic extension of the max-operation.
2 Main Result: Case of Min Operation

Definition 1. Let \((S, \leq)\) be a partially ordered set with the largest element 1 that contains two elements \(a\) and \(b\) for which \(a < b < 1\). Let \& be a commutative operation on the set \(S\) for which \(a \& 1 = a\) for all \(a\). We say that the order \(\leq\) is lexicographic if for all \(a_1 \leq b_1\) and \(a_2 \leq b_2\), we have \(a_1 \& b_1 \leq a_2 \& b_2\) if and only if:

- either \(a_1 < a_2\),

- or \(a_1 = a_2\) and \(b_1 \leq b_2\).

Proposition 1. When the order is lexicographic, the operation \& is not associative.

Proof. Let us consider the elements \(a < b < 1\) whose existence is guaranteed by the definition of lexicographic order. Then, by this definition, for \(a_1 = a_2 = a\), \(b_1 = b\), and \(b_2 = 1\), we get

\[ a \& b < a \& 1. \tag{1} \]

From \(a < b\) and from the fact that \(a \& 1 = a\), we conclude that

\[ a \& 1 = a < b. \tag{2} \]

Now, for \(a_1 = a \& b\), \(a_2 = a \& 1\), \(b_1 = 1\), and \(b_2 = b\):

- we have \(a_1 \leq b_1\) – since 1 is the largest element,

- we have \(a_2 \leq b_2\) – by formula (2), and

- we have \(a_1 < a_2\) – by formula (1).

So, since the order is lexicographic, we can conclude that \(a_1 \& b_1 < a_2 \& b_2\), i.e., that

\[ (a \& b) \& 1 < (a \& 1) \& b, \tag{3} \]

while by associativity and commutativity, we would have \((a \& b) \& 1 = (a \& 1) \& b\). Thus, the operation \& is not associative.

The proposition is proven.

3 Main Result: Case of Max Operation

Definition 2. Let \((S, \leq)\) be a partially ordered set with the smallest element 0 that contains two elements \(a\) and \(b\) for which \(0 < a < b\). Let \(\lor\) be a commutative operation on the set \(S\) for which \(a \lor 0 = a\) for all \(a\). We say that the order \(\leq\) is lexicographic if for all \(a_1 \leq b_1\) and \(a_2 \leq b_2\), we have \(a_1 \lor b_1 \leq a_2 \lor b_2\) if and only if:

- either \(b_1 < b_2\),
• or \( b_1 = b_2 \) and \( a_1 \leq a_2 \).

**Proposition 2.** When the order is lexicographic, the operation \( \lor \) is not associative.

**Proof.** Let us consider the elements \( 0 < a < b \) whose existence is guaranteed by the definition of lexicographic order. Then, by this definition, for \( a_1 = 0 < a_2 = a \), and \( b_1 = b_2 = b \), we get

\[
0 \lor b < a \lor b. \tag{4}
\]

From \( a < b \) and from the fact that \( 0 \lor b = b \), we conclude that

\[
a < 0 \lor b = b. \tag{5}
\]

Now, for \( a_1 = a, a_2 = 0, b_1 = 0 \lor b, \) and \( b_2 = a \lor b \):

• we have \( a_1 \leq b_1 \) – by formula (5),

• we have \( a_2 \leq b_2 \) – since 0 is the smallest element, and

• we have \( b_1 < b_2 \) – by formula (4).

So, since the order is lexicographic, we can conclude that \( a_1 \lor b_1 < a_2 \lor b_2 \), i.e., that

\[
a \lor (0 \lor b) < 0 \lor (a \lor b), \tag{6}
\]

while by associativity and commutativity, we would have \( a \lor (0 \lor b) = 0 \lor (a \lor b) \).

Thus, the operation \( \lor \) is not associative.

The proposition is proven.

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**References**


